Qualifying Exam
GEOMETRY-TOPOLOGY
Spring 2006

All ten problems have equal value.

Part I: Differentiable Manifolds

1. (i) Suppose that M is a closed (that is, compact and without boundary) smooth m-manifold. Show that there is a smooth embedding $f\colon M \hookrightarrow \mathbb{R}^n$ for sufficiently large n.
(ii) Adapt/extend your argument to show that if $g\colon M \to \mathbb{R}^n$ is a given *continuous* map, then the smooth embedding (of part (i))$f\colon M \hookrightarrow \mathbb{R}^n$ can be chosen to be arbitrarily (pointwise) close to g (again for sufficiently-large-but-fixed n).

2. Let $\omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$ be a 2-form defined on $\mathbb{R}^{2n}$, where $(x_1, y_1, \cdots, x_n, y_n)$ are the coordinates of $\mathbb{R}^{2n}$.
(i) Show that as a bilinear form defined on $\mathbb{R}^{2n}$, ω is non-degenerate.
(ii) Let $f\colon \mathbb{R}^{2n} \to \mathbb{R}^1$ be smooth. Show that there is a unique vector field X_f on $\mathbb{R}^{2n}$ such that for any vector field Y on $\mathbb{R}^{2n}$, $df(Y) = \omega(X_f, Y)$.
(iii) Use the formula $\mathcal{L}_X = i_X \circ d + d \circ i_X$ to compute the Lie derivative $\mathcal{L}_{X_f}\omega$. Here $i_X : \Omega^k(\mathbb{R}^{2n}) \to \Omega^{k-1}(\mathbb{R}^{2n})$ denotes the *interior product* (or *contraction*) defined by

$$i_X(\eta)(Y_1, \ldots, Y_{k-1}) := \eta(X, Y_1, \ldots, Y_{k-1}).$$

3. Let $\omega = (x_1^2 + \cdots + x_n^2)^{-\frac{n}{2}} \sum_{i=1}^n (-1)^{i-1} x_i \, dx_1 \wedge dx_2 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_n$
be an $(n-1)$-form defined on $\mathbb{R}^n - \{\mathbf{0}\}$.
(i) Suppose f is a smooth map from a closed oriented manifold M of dimension $n-1$ to $\mathbb{R}^n - \{\mathbf{0}\}$. Show that $\int_M f^*\omega$ only depends on the homotopy class of f.
(ii) Find the possible values of the integrals in (i) in the case that $n = 3$ and $M = S^2$.

4. Suppose M and N are two smooth manifolds of positive dimensions m and n respectively, and f is a smooth map from M to N.
(i) If $m < n$, is it possible that f is surjective? Justify appropriately your answer.
(ii) If $m \geq n$, must some point-inverse $f^{-1}(y)$ be a smooth $(m-n)$-dimensional submanifold of M? Justify appropriately your answer.
(iii) Show that some point-inverse $f^{-1}(y)$ can be homeomorphic to a Cantor set.
Hint: The model case is where $f\colon \mathbb{R}^1 \to [0, \infty) \subset \mathbb{R}^1$, with $f^{-1}(0) =$ Cantor set. The desired f can be constructed as a suitable limit of sums of C^∞ bump functions. Supply details, extending your argument to the general case (arbitrary M and N) of the question.

5. (i) Let $f\colon \mathbb{R}^n \to \mathbb{R}^1$ be a smooth function. Show that there are smooth functions $g_1, \cdots, g_n$ from $\mathbb{R}^n$ to $\mathbb{R}^1$ such that $f(\boldsymbol{x}) = f(\mathbf{0}) + \sum_{j=1}^n g_j(\boldsymbol{x})x_j$ and $g_j(\mathbf{0}) = \frac{\partial f}{\partial x_j}(\mathbf{0})$, where $\boldsymbol{x} = (x_1, \ldots, x_j, \ldots, x_n)$. (Hint: Recall that such g_j's can be defined using integrals.)

(ii) Let F be a diffeomorphism of $\mathbb{R}^n$ to itself. Use (i) to find a smooth isotopy (= a smooth homotopy which is a diffeomorphism at each fixed time of the homotopy) between F and $DF(\mathbf{0})$. (Hint: To find the isotopy F_t, $0 \le t \le 1$, you may assume that $F(\mathbf{0}) = \mathbf{0}$ (justify this) and then define $F_t(\boldsymbol{x}) = F(t\boldsymbol{x})/t$ for $0 < t \le 1$.)

Part II: Algebraic Topology

6. (i) Define what it means for two spaces X and Y to be *homotopically equivalent* (equivalently, to have the *same homotopy type*).

(ii) Define what it means for a space W to be *contractible.* (If you wish, you may reference your definition in part (i).)

(iii) Suppose that X is a manifold and W is an arbitrary contractible space. Show that X and the wedge $Y := X \underset{x_0 \sim w_0}{\vee} W$ are homotopically equivalent. (Here $x_0 \in X$ and $w_0 \in W$ are (arbitrary) points, and $X \underset{x_0 \sim w_0}{\vee} W$ is the one-point union of X and W at these points.)

7. (i) Define what it means for a map $p\colon X \to Y$ to have the (*unique*) *homotopy lifting property* (= HLP here), equivalently known as the (*unique*) *covering homotopy property* (= CHP). Recall (partly to establish some notation) that the definition begins:
$p\colon X \to Y$ has the HLP if, given any space W and homotopy $F\colon W \times [0,1] \to Y$ such that ....

(ii) Show that a covering map $p\colon X \to Y$ has the HLP for the special case where W is a point.

8. (i) Suppose that $X = U \cup V$ is the union of two open subsets U and V whose intersection is path-connected, and let $x_0 \in U \cap V$. State (carefully) the (Seifert -)van Kampen Theorem for these data, relating $\pi_1(X, x_0)$ to $\pi_1(U, x_0)$, $\pi_1(V, x_0)$ and $\pi_1(U \cap V, x_0)$.

(ii) Prove the special case of this theorem which asserts that the natural homomorphism $\pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0)$ is an epimorphism.

9. Suppose that $\mathcal{A} = \{\partial_n^A : A_n \to A_{n-1} \mid n \geq 0\}$ is a chain complex (with it understood that $A_{-1} = 0$).

(i) Define the nth homology group $H_n(\mathcal{A})$.

(ii) Suppose that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are chain complexes, with (connecting) homomorphisms $\alpha = \{\alpha_n \colon A_n \to B_n\}$ and $\beta = \{\beta_n \colon B_n \to C_n\}$ such that

$$\begin{array}{ccccccccc}
 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A_{n+1} & \xrightarrow[\partial^A_{n+1}]{} & A_n & \xrightarrow[\partial^A_n]{} & A_{n-1} & \longrightarrow & \cdots \\
\downarrow & & \downarrow{\scriptstyle \alpha_{n+1}} & & \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \alpha_{n-1}} & & \\
\cdots & \longrightarrow & B_{n+1} & \xrightarrow[\partial^B_{n+1}]{} & B_n & \xrightarrow[\partial^B_n]{} & B_{n-1} & \longrightarrow & \cdots \\
\downarrow & & \downarrow{\scriptstyle \beta_{n+1}} & & \downarrow{\scriptstyle \beta_n} & & \downarrow{\scriptstyle \beta_{n-1}} & & \\
\cdots & \longrightarrow & C_{n+1} & \xrightarrow[\partial^C_{n+1}]{} & C_n & \xrightarrow[\partial^C_n]{} & C_{n-1} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & &
\end{array}$$

all squares commute and all (vertical) columns are exact. Show how to define the *boundary homomorphism* $\partial_n \colon H_n(\mathcal{C}) \to H_{n-1}(\mathcal{A})$, and justify that it is well-defined.

(iii) Define and prove *exactness* at $H_{n-1}(\mathcal{A})$ (for the long exact sequence $\cdots \to H_n(\mathcal{C}) \to H_{n-1}(\mathcal{A}) \to H_{n-1}(\mathcal{B}) \to \cdots$).

10. Let S^p and S^q be (standard) spheres of (arbitrary) dimensions $p \geq 0$ and $q \geq 0$. Compute the homology groups $H_n(S^p \times S^q)$ for all $n \geq 0$. You may use any reasonable method (e.g. Mayer-Vietoris, or cellular homology), as long as you present your argument with suitable completeness and clarity.