Each problem is worth 10 points. In order to pass this examination, you must demonstrate proficiency in both parts, differentiable manifolds and algebraic topology. In particular, if you score fewer than 20 points on one part, you will not pass no matter how well you do on the other part.

Part I: Differentiable Manifolds

1. Let M_n be the linear space of all $n \times n$ of real matrices and S_n be the subspace of all $n \times n$ symmetric matrices. Consider the smooth map $\psi \colon M_n \to S_n$ defined by $\psi(A) = A^t A - I_n$ for $A \in M_n$, where A^t is the transpose of A and I_n is the identity matrix. (a) (5 points) Show that $0 \in S_n$ is a regular value of ψ. (b) (5 points) Use (a) to show that the group $O(n)$ of all orthogonal $n \times n$-matrices is a compact Lie group.

2. Let $f \colon \mathcal{R}^{n+1} \to \mathcal{R}$ be a smooth function defined on $\mathcal{R}^{n+1}$. Assume that $0 \in \mathcal{R}$ is a regular value of f. (a) (3 points) Show that $M = f^{-1}(0)$ is a smooth submanifold of $\mathcal{R}^{n+1}$. (b) (3 points) Show that M has a non-vanishing normal field, i.e. there is a smooth map $N \colon M \to \mathcal{R}^{n+1}$ with $N(x) \neq 0$ for any $x \in M$ such that $< N(x), v > = 0$ for any tangent vector V of M at any point x in M. (c) (4 points) Use (b) to show that $M \times S^1$ is parallelizable, i.e, there exist $n+1$ linearly independent vector fields defined on $M \times S^1$.

3. Let $S^{2n-1} = \{z = (z_1, \cdots, z_n) \mid |z_1|^2 + \cdots |z_n|^2 = 1\}$ be the unit sphere in $\mathcal{R}^{2n}$ identified with $\mathcal{C}^n$, where $z_i, i = 1, \cdots, n$, are the complex coordinates of a point z in $\mathcal{C}^n$. The cyclic group $\mathcal{Z}_p = \mathcal{Z}/(p)$ of order p acts on S^{2n-1} freely by the formula: $\phi(z) = e^{\frac{2\pi i}{p}} z$, where ϕ is the generator of $\mathcal{Z}_p$. Let M be the quotient space of S^{2n-1} under this action. (a) (5 points) Show that any closed 1-form on S^n is exact if $n > 1$. (b) (5 points) Show that the same conclusion as (a) is true for $M = S^{2n-1}/\mathcal{Z}_p$ if $n > 1$.

4. Let M be the space defined in Problem 3 with $n > 1$. (a) (5 points) Let $f \colon M \to T^m$ be a continuous map, where $T^m =$

$S^1 \times \cdots \times S^1$ is the standard m-dimensional torus. Show that f is homotopically trivial. (b) (5 points) Construct explicitly a homotopically non-trivial map $f \colon M \to S^{2n-1}$.

5. Let D be a bounded domain in $\mathcal{R}^n$ with a smooth boundary S and let X be a smooth vector field defined on $\mathcal{R}^n$. (a) (5 points) Let $\omega = dx_1 \cdots dx_n$. Show that the Lie derivative $\mathcal{L}_X \omega = div(X)\omega$. (b) (5 points) Use Stokes' theorem to show that

$$\int_D div(X)\,\omega = \pm \int_S <X, N> \, dS.$$

Here N is the outer unit normal vector field along S, $<X, N>$ is the Euclidean inner product of X and N, and "dS" is the "area" form on S. Explain carefully the geometrical meaning of the term "dS".

Part II: Algebraic Topology

6. Let $f, g \colon X \to Y$ be homotopic maps of not necessarily connected spaces. Prove the following for the specific dimensions given; do not refer to any general result. (a) (3 points) Prove that $f_{*,0} = g_{*,0} \colon H_0(X) \to H_0(Y)$. (a) (7 points) Prove that $f_{*,1} = g_{*,1} \colon H_1(X) \to H_1(Y)$.

7. Let $x_1, x_2 \in S^1$, the circle, and define subsets of the torus $T = S^1 \times S^1$ as follows: $A = (S^1 \times \{x_1\}) \cup (\{x_1\} \times S^1)$ (a figure-eight) and $B = (S^1 \times \{x_1\}) \cup (S^1 \times \{x_2\})$ (two disjoint circles). Assume the homology of T is known. (a) (4 points) Calculate the homology of the pair (T, A). (b) (6 points) Calculate the homology of the pair (T, B).

8. (a) (3 points) Prove that the 2-sphere S^2 is simply-connected. (b) (7 points) Each compact orientable surface S_g can be obtained by adding g handles to S^2, that is, g pairs of open discs are removed from S^2 and g copies of $S^1 \times [0, 1]$ are attached to pairs of bounding circles. Prove that S^2 is the only compact orientable surface that is simply-connected.

9. (a) (2 points) Define the *degree deg(f)* of a map $f \colon S^2 \to S^2$ and prove that this is well-defined, that is, independent of any choices required for the definition. (b) (8 points) Give a detailed proof

that, given an integer k, there exists a map $f_k\colon S^2 \to S^2$ such that $deg(f) = k$.

10. Let X be a path-connected space and let $p\colon \widetilde{X} \to X$ be a normal n-sheeted covering space, that is, each $p^{-1}(x)$ consists of n points and, for each pair of such points, there is a deck transformation taking one to the other. (a) (5 points) Prove that if $\widetilde{X}$ is path-connected, then the order of the fundamental group $\pi_1(X, x_0)$ is at least n. (b) (5 points) Prove that if X is simply-connected, then $\widetilde{X}$ has n path components.