Qualifying Exam
GEOMETRY-TOPOLOGY
March 2007

Each problem is worth 10 points. In order to pass this examination, you must demonstrate proficiency in both parts, differential manifolds and algebraic topology. In particular, if you score fewer than 20 points on one part, you will not pass no matter how well you do on the other part.

Part I: Differentiable Manifolds

1, Let M be a smooth three-dimensional manifold and let α be a 1-form on M such that $\alpha \wedge d\alpha \neq 0$ at every point of M.

(i) Let $H = ker\, \alpha \subseteq TM$. Show that H is a two-dimensional subbundle of TM that is not integrable. (Hint: Use the formula

$$d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y])$$

where X, Y are two arbitrary vector fields.)

(ii) Show that there is a unique vector field V such that

$$(a)\,\alpha(V) = 1 \quad (b) < V > \oplus H = TM \quad (c)\, d\alpha(V,W) = 0$$

for any vector field W. Here $< V >$ is the line field generated by V.

2. Suppose $f: S^1 \to \mathbf{R^2}$ and $g: S^1 \to \mathbf{R^2}$ are smooth embeddings. Let

$$M = \{(a, b, \vec{v}) \in S^1 \times S^1 \times \mathbf{R^2}: f(a) - g(b) = \vec{v}\}.$$

Show that M is a compact submanifold of $S^1 \times S^1 \times \mathbf{R^2}$. Let $\pi: M \to \mathbf{R^2}$ be the projection $\pi(a, b, \vec{v}) = \vec{v}$. Apply Sard's Theorem to π and deduce that for almost every $\vec{v} \in \mathbf{R^2}$, $f(S^1)$ is transverse to $g(S^1) + \vec{v}$.

3. Let $f: \mathbf{R^n} \to \mathbf{R}$ be a smooth function and let $x \in \mathbf{R^n}$ be a critical point of f. The Hessian $H(f)_x$ at x is a bilinear function $H(f)_x: T_x\mathbf{R^n} \times T_x\mathbf{R^n} \to \mathbf{R}$ defined as follows: given vectors V_1 and V_2 in $T_x\mathbf{R^n}$, extend V_2 to a vector field $\widetilde{V}_2$ near x and define

$$H(f)_x(V_1, V_2) = D_{V_1} D_{\widetilde{V}_2} f.$$

(i) Show that $H(f)_x(V_1, V_2)$ is independent of the extension $\widetilde{V}_2$.

(ii) Show that $H(f)_x(V_1, V_2) = H(f)_x(V_2, V_1)$.

4. Suppose M is a compact, connected n-manifold, α is a p-form and β is an $n-p-1$-form. The boundary ∂M has two components, $\partial_0 M$ and $\partial_1 M$. Let i_0 and i_1 be the inclusions of $\partial_0 M$ and $\partial_1 M$ into M. Given that $i_0^*\alpha = 0$ and $i_1^*\beta = 0$, show that

$$\int_M d\alpha \wedge \beta = (-1)^{p+1} \int_M \alpha \wedge d\beta.$$

5. Show that $T^2 \times S^2$ is parallelizable.

Part II: Algebraic Topology

6. The *cone* CA of a space A is obtained from $A \times I$ by identifying $A \times \{1\}$ to a point p. Prove that, if (X, A) a topological pair, then $H_k(X, A)$ is isomorphic to $\widetilde{H}_k(X \cup CA)$ for all k, where $\widetilde{H}$ denotes reduced homology.

7. (a) Let X be a path-connected, locally path-connected and simply-connected space. Prove that if $f, g: X \to S^1$ are maps, then they are homotopic. (b) Represent S^1 as the complex numbers of norm one and define $p_k: S^1 \to S^1$ by $p_k(z) = z^k$. Prove that a map $f: S^1 \to S^1$ such that $f(1) = 1$ is homotopic to p_k, for some k, by a homotopy $F: S^1 \times I \to S^1$ such that $F(1, t) = 1$ for all t.

8. Calculate the homology of the complement of a finite set of $n \geq 1$ points in $\mathbf{R}^3$ and give a convincing argument that your calculation is correct.

9. Let $A \subseteq X$ and let $i: A \to X$ be inclusion inducing $i_*: \pi_1(A, x_0) \to \pi_1(X, x_0)$. Given a covering space $p: \widetilde{X} \to X$, prove that the kernel of i_* is contained in the image of $p_{A*}: \pi_1(p^{-1}(A), \tilde{x}_0) \to \pi_1(A, x_0)$, where $p_A: p^{-1}(A) \to A$ denotes the restriction of p.

10. Suppose $B \subseteq A \subseteq X$ and $j: (X, B) \to (X, A)$ is inclusion. Define $\partial: H_k(X, A) \to H_{k-1}(A, B)$ so that the kernel of ∂ is equal to the image of $j_*: H_k(X, B) \to H_k(X, A)$ and prove that it has this property.