Qualifying Exam
Geometry/Topology
September 2008

Each problem is worth 10 points. In order to pass this examination, you must demonstrate proficiency in both parts, manifold theory and algebraic topology. In particular, if your score is fewer than 20 points on one part, you will not pass no matter how well you do on the other part.

Part I: Manifold Theory

1. Let $G(k,n)$ be the collection of all k-dimensional linear subspaces in $\mathbf{R}^n$.

(i) Define a natural topological and smooth structure on $G(k,n)$, and show that with respect the structures you defined, $G(k,n)$ is a compact smooth manifold.

(ii) Show that $G(k,n)$ is diffeomorphic to $G(n-k,n)$.

2. Let M and N be two smooth manifolds, and $f : M \to N$ be a smooth map. Assume that $df_x : T_xM \to T_{f(x)}N$ is surjective for all x in M and that the inverse image $f^{-1}(y)$ is compact for all y in N.

(i) Show that for any x in M there is an open neighborhood U of x such that $f^{-1}(U)$ is diffeomorphic to $U \times f^{-1}(x)$.

(ii) Assume further that N is connected, can you take U to be N in (i) ? (Justify your answer)

3. Let M be a connected smooth manifold. Show that for any two points x and y in M there is a diffeomorphism f of M such that $f(x) = y$.

4. Let $\theta = \Sigma_{i=1}^n(x_i dy_i - y_i dx_i)$ be a 1-form defined on $\mathbf{R}^{2n}$, where $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ are the coordinates of $\mathbf{R}^{2n}$. Consider the $2n-1$ dimensional distribution $D = ker\theta$. Is D integrable? (Justify your answer)

5. Let D be a bounded domain in $\mathbf{R}^n$ with a smooth boundary S, $j : S \to \mathbf{R}^n$ be the inclusion map and X be a smooth vector field defined on $\mathbf{R}^n$.

(i) Denote the standard volume form $dx_1 \wedge \cdots \wedge dx_n$ by ω. Show that $j^*(i_X\omega) = < X, N > dS$, where N is the outer unit normal vector field along S, $< X, N >$ is the Euclidean inner product of X and N. Here $i_X\omega$ is the contraction of ω along X, dS is the "area" form on S. Explain carefully the definition and geometrical meaning of the term "dS".

(ii) Use (i) and Stokes theorem to show that

$$\int_D \mathcal{L}_X \omega = \int_S < X, N > dS.$$

Here $\mathcal{L}_X \omega$ is the Lie derivative of ω along X.

Part II: Algebraic Topology

6. Find $\pi_1(T^2 \setminus \{k\ pts\})$, where T^2 is the two-dimensional torus.

7. Find the homology groups $H_i(\Delta_n^{(k)}), i = 0, 1, \cdots, k$. Here $\Delta_n^{(k)}$ is the k-skeleton of the n-simplex Δ_n with $k \leq n$.

8. Let G be a topological group with the identity element e. For any two continuous loops γ_1 and $\gamma_2 : S^1 \to G$ sending $1 \in S^1$ to $e \in G$, define $\gamma_1 * \gamma_2 : S^1 \to G$ by $\gamma_1 * \gamma_2(t) = \gamma_1(t) \circ \gamma_2(t)$ for $t \in S^1$. Here $\circ$ is the product operation in G.

(i) Show that the product $*$ so defined induces a product structure on $\pi_1(G, e)$ and that this new product on $\pi_1(G, e)$ is the same as the usual one.

(ii) Is $\pi_1(G, e)$ commutative? (Justify your answer)

9. (i) Show that any continuous map $f : S^2 \to T^2$ is null-homotopic.

(ii) Show that there exists a continuous map $f : T^2 \to S^2$ which is not null-homotopic.

10. Let A and B be two chain complexes with boundary operators ∂_A and ∂_B respectively, and $f : A \to B$ be a chain map. Define a new chain complex C whose ith chain group is $C_i = A_i \oplus B_{i+1}$ and whose boundary operator ∂_C is defined by $\partial_C(a, b) = (\partial_A(a), \partial_B(b) + (-1)^{deg(a)} f(a))$ for any $(a, b) \in C_i$. Here A_i and B_i are the ith chain groups of A and B respectively.

(i) Show that C so defined is indeed a chain complex and that there is a short exact sequence of chain complexes:

$$0 \to B \to C \to A \to 0$$

sending B_{i+1} to C_i and C_i to A_i.

(ii) Write down the long exact sequence of the homology groups associated to the short exact sequence in (i). What is the connecting boundary map in the long exact sequence?

(iii) Let $(f_*)_i : H_i(A) \to H_i(B)$ be the induced map of f on the ith homology groups. Show that $(f_*)_i : H_i(A) \to H_i(B)$ is an isomorphism for all i if and only if $H_i(C) = 0$ for all i.