GIT S08

1. Let M and N be smooth (C^∞) manifolds, not necessarily of the same dimension, and $F: M \to N$ be a smooth map.
   (a) Define the map F^* of p-forms on N to p-forms on M (p=0,1,2,...).
   (b) Prove that, if ω is a p-form on N, then $F^*(d_N\omega)=d_M(F^*\omega)$.

2. Let M be a C^∞ manifold and X a C^∞ vector field on M.
   (a) Suppose $X(p)\neq 0$ for some particular $p\in M$. Show , using the flow of X, that there is a neighborhood U of p and a coordinate system$(x_1,\ldots,x_n)$ on U with $X=\partial/\partial x_1$ on U.
   (b) Use part (a) to prove that if Y is another C^∞ vector field on M with [X, Y]=0 everywhere on M, then $\varphi_s(\psi_t(p))=\psi_t(\varphi_s(p))$ for all s, t with $|t|$ and $|s|$ sufficiently small, where φ, ψ are the flows of X and Y respectively. [Suggestion: Write Y near p in the coordinate system of part (a)].

3. Gauss's Divergence Theorem asserts that if U is a bounded open set in R^3 with smooth boundary and if X is a smooth vector field defined in a neighborhood of the closure of U, then $$\iiint_U \text{divergence}(X)\, d(\text{vol}) = \iint_{\partial U} X\cdot N\, d(\text{area})$$
where N is the exterior unit normal to ∂U. Show how the Divergence Theorem follows from Stokes Theorem for differential forms on manifolds with boundary.

4. (a)Let θ be a 1-form on S^2 with $d\theta=0$. Construct a function f on S^2 with $df = \theta$.

   (b)Let θ be a 1-form on $S^1 \times (0, 1)$ with $d\theta=0$. Show that there is a function $f: S^1 \times (0, 1)\to R$ with $df = \theta$ if and only if $\int_{S^1\times \frac{1}{2}} \theta = 0$.

   (c) Use part (b) to show that if ω is a 2-form on S^2 with $\int_{S^2}\omega = 0$ then there is a 1-form θ on S^2 with $d\theta = \omega$. [Suggestion: You may assume the Poincaré Lemma so that $\omega = d\theta_1$ on S^2 - {South pole} and $\omega = d\theta_2$ on S^2 - {North pole} . Use Stokes theorem to show $\theta_1 - \theta_2$ satisfies the integral condition of part (b)].

5. Let SO(3) = the set of all 3x3 matrices A with AA^t = identity (orthogonal matrices) and determinant of A = 1. Also, for each 3x3 matrix B, let
$$\exp(B) = I + B + \left(\frac{B^2}{2!}\right) + \left(\frac{B^3}{3!}\right) + \ldots .$$
   (a) Prove that the infinite series for exp(B) converges for each 3x3 matrix B, so that exp is a map from the space of 3x3 matrices to itself.

You may *assume* from here on that this map is smooth and that the series can be differentiated term by term to give the differential of the mapping.

   (b) Show that the map exp is injective on some neighborhood of the 0 matrix in the space of all 3x3 matrices. [Suggestion: Inverse function theorem].

   (c) Prove that exp(B) is in SO(3) if B satisfies $B^t = -B$ (B is "anti-symmetric") .

   (d) Show that the mapping exp restricted to the vector space of 3x3 anti-symmetric matrices is a surjective (onto) map from some neighborhood of the 0 matrix to a neighborhood of the identity matrix in SO(3). [Suggestion: Note that every element of SO(3) is a rotation around an axis, so check this case.]

(e) Discuss how to combine parts (b), (c), and (d) to give coordinate charts on SO(3) and thus to make SO(3) a differentiable manifold.

6. Let M and N be two compact, oriented manifolds of the same dimension. And let ω be a nowhere vanishing n-form on N with $\int_N \omega = 1$. Let F: M $\rightarrow$ N be a smooth map.

(a) Set $\deg_\omega F = \int_M F^*\omega$. Show that $\deg_\omega F$ is independent of the choice of ω.

[You may assume deRham's Theorem]. We shall call the common value the degree of F.

(b) Show that there is a smooth map from $S^2 \times S^2$ to S^4 of degree 1.

(c) Show that no map from S^4 to $S^2 \times S^2$ has degree 1.

7. Describe carefully the basic algebraic construction of algebraic topology, namely, how to go from a short exact sequence of chain complexes to a long exact sequence in homology. Give explicitly, in particular, the construction of the "connecting homomorphism", the map where the dimension drops, and prove exactness at its image, that is, prove that the image of the connecting homorphism = the kernel of the map that follows it. [You need not prove exactness of the long exact sequence elsewhere].

8(a)Prove that S^n is simply connected if $n > 1$.

(b)Prove that $\pi_1(RP^n) = Z_2$, n>1.

(c)Prove that RP^n is orientable if n is odd (n >1).

9. Find by any method the homology groups of RP^n with integer coefficients.

10 (a)Define complex projective space CP^n .

(b)Show that CP^n is compact.

(c)Show that $CP^1 = S^2$ (homeomorphic is enough).

(d) Show that CP^n is simply connected.

(e) Find the homology of CP^n (integer coefficients). [Any method will do. But cell complex decomposition is the easiest].