Qualifying Exam
GEOMETRY-TOPOLOGY
March 2009

Instructions: Do any ten of the following twelve problems. Please do not turn in work on more than ten problems and label each problem carefully by its number. Start each problem on a new page.

1. (a) Show that a closed 1-form θ on $S^1 \times (-1,1)$ is dF for some function $F: S^1 \times (-1,1) \to \mathbf{R}$ if and only if $\int_{S^1} i^*\theta = 0$ where $i: S^1 \to S^1 \times (-1,1)$ is defined by $i(p) = (p,0)$ for $p \in S^1$. (b) Show that a 2-form ω on S^2 is $d\theta$ for some 1-form θ on S^1 if and only if $\int_{S^2} \omega = 0$.

2. Suppose that M, N are connected C^∞ manifolds of the same dimension $n \geq 1$ and $F: M \to N$ is a C^∞ map such that $dF: T_pM \to T_{F(p)}N$ is surjective for each $p \in M$. (a) Prove that if M is compact, then F is onto and F is a covering map. (b) Find an example of such an everywhere nonsingular equidimensional map where N is compact, F is onto, $F^{-1}(p)$ is finite for each $p \in N$, but F is not a covering map. [A clearly explained pictorial version of F will be acceptable; you do not need to have a "formula" for F.]

3. (a) Suppose that M is a C^∞ connected manifold. Prove that, given an open subset U of M and a finite set of points $p_1, p_2, \ldots, p_k$ in M, there is a diffeomorphism $F: M \to M$ such that $f(\{p_1, p_2, \ldots, p_k\}) \subset U$. [Suggestion: Construct F one point at a time.] (b) Use part (a) to show that if M is compact and the Euler characteristic $\chi(M) = 0$, then there is a vector field on M which vanishes nowhere. You may assume that if a vector field has isolated zeros, then the sum of the indices at the zero points equals $\chi(M)$.

4. A smooth vector field V on $\mathbf{R^3}$ is said to be "gradient-like" if, for each $p \in \mathbf{R^3}$, there is a neighborhood U_p of p and a function $\lambda_p: U_p \to \mathbf{R} - \{\mathbf{0}\}$ such that $\lambda_p V$ on U_p is the gradient of some C^∞ function on U_p. Suppose V is nowhere zero on $\mathbf{R^3}$. Then show that V is gradient-like if and only if $curl\, V$ is perpendicular to V at each point of $\mathbf{R^3}$

5. Suppose that M is a compact C^∞ manifold of dimension n. (a) Show that there is a positive integer k such that there is an immersion $F: M \to \mathbf{R^k}$. (b) Show that if $k > 2n$, there is a $(k-1)$-dimensional subspace H of $\mathbf{R^k}$ such that $P \circ F$ is an immersion, where $P: \mathbf{R^k} \to H$ is orthogonal projection.

6. Let $Gl^+(n, \mathbf{R})$ be the set of $n \times n$ matrices with determinant > 0. Note that $Gl^+(n, \mathbf{R})$ can be considered to be a subset of $\mathbf{R^{n^2}}$ and this subset is open. (a) Prove that $Sl^+(n, \mathbf{R}) = \{A \in Gl^+(n, \mathbf{R}) : \det A = 1\}$ is a submanifold. (b) Identify the tangent space of $Sl^+(n, \mathbf{R})$ at the identity matrix I_n. (c) Prove that, for every $n \times n$ matrix B, the series $I_n + B + \frac{1}{2}B^2 + \frac{1}{3!}B^3 + \cdots + \frac{1}{n!}B^n \cdots$ converges to some $n \times n$ matrix. Notation: this sum $= e^B$. (d) Prove that if $e^{tB} \in Sl^+(n, \mathbf{R})$ for all $t \in \mathbf{R}$,

then $trace\, B = 0$. (e) Prove that if $trace\, B = 0$, then $e^B \in Sl^+(n, \mathbf{R})$. [Suggestion: Use one-parameter subgroups or note that it suffices to treat complex-diagonable B since such are dense.]

7. (a) Define complex projective space $\mathbf{CP^n}$. (b) Calculate the homology of $\mathbf{CP^n}$. Any systematic method such as Mayer-Vietoris or cellular homology is acceptable.

8. Let $p: E \to B$ be a covering space and $f: X \to B$ a map. Define $E^* = \{(x, e) \in X \times B: f(x) = p(e)\}$. Prove that $q: E^* \to X$ defined by $q(x, e) = x$ is a covering space.

9. (a) Explain carefully and concretely what it means for two (smooth) maps of S^1 into $\mathbf{R^2}$ to be transversal. (b) Do the same for maps of S^1 into $\mathbf{R^3}$. (c) Explain what it means for transversal maps to be "generic" and prove that they are indeed generic in the cases of 9(a) and 9(b).

10. Let M be the 3-manifold with boundary obtained as the union of the two-holed torus in 3-space and the bounded component of its complement. Let X be the space obtained from M by deleting k points from the interior of M. (a) Calculate the fundamental group of X. (b) Calculate the homology of X.

11. Let P be a finite polyhedron. (a) Define the Euler characteristic $\chi(P)$ of P. (b) Prove that if P_1, P_2 are subpolyhedra of P such that $P_1 \cap P_2$ is a point and $P_1 \cup P_2 = P$, then $\chi(P) = \chi(P_1) + \chi(P_2) - 1$. (c) Suppose that $p: E \to P$ is an n-sheeted covering space of P, that is $p^{-1}(x)$ is n points for each $x \in P$. Prove that $\chi(E) = n\chi(P)$

12. Let $f: T \to T = S^1 \times S^1$ be a map of the torus inducing $f_\pi: \pi_1(T) \to \pi_1(T) = \mathbf{Z} \oplus \mathbf{Z}$ and let F be a matrix prepresenting f_π. Prove that the determinant of F equals the degree of the map of the map f.