QUALIFYING EXAM
Geometry/Topology
September 2010

Each problem is worth 10 points. Justify your answers carefully.

1. Let M be a connected smooth manifold. Show that for any two non-zero tangent vectors v_1 at point x_1 and v_2 at point x_2, there is a diffeomorphism $\phi : M \to M$ such that $\phi(x_1) = x_2$ and $d\phi(v_1) = v_2$.

2. Let X and Y be submanifolds of $\mathbb{R}^n$. Prove that for almost every $a \in \mathbb{R}^n$, the translate $X + a$ intersects Y transversely.

3. Let $M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$ be the space of $n \times n$ matrices with real coefficients.

(a) Show that

$$SL(n, \mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) | \det(A) = 1\}$$

is a smooth submanifold of $M_{n\times n}(\mathbb{R})$.

(b) Identify the tangent space to $SL(n, \mathbb{R})$ at the identity matrix I_n.

(c) Show that $SL(n, \mathbb{R})$ has trivial Euler characteristic.

4. (a) Let $f_i : M \to N, i = 0, 1$, be two smooth maps between smooth manifolds M and N, and $f_i^* : \Omega^*(N) \to \Omega^*(M), i = 0, 1$, be the induced chain maps between the respective de Rham complexes. Define the notion of chain homotopy between f_0^* and f_1^*. Here the co-boundary operators on the de Rham complexes are the exterior derivatives.

(b) Let X be a smooth vector field on a compact smooth manifold M, and let $\phi_t : M \to M$ be the flow generated by X at time t, i.e. the solution of the differential equation $\frac{d\phi_t}{dt}(x) = X(\phi_t(x))$ with initial condition $\phi_0(x) = x$. Find an explicit chain homotopy between the chain maps ϕ_0^* and ϕ_1^*, where $\phi_i^*, i = 0, 1$, are the induced chain maps from $\Omega^*(M)$ to itself.

Hint: Use the formula that for any differential form ω and vector field X, the Lie derivative $\mathcal{L}_X\omega = d \circ i_X\omega + i_X \circ d\omega$. Here i_X is the contraction with respect to X.

5. Let $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + \cdots + dx_{2n-1} \wedge dx_{2n}$ be a 2-form on $\mathbb{R}^{2n}$, where $(x_1, x_2, \ldots, x_{2n})$ are the standard coordinates on $\mathbb{R}^{2n}$. Define an S^1-action on $\mathbb{R}^{2n}$ as follows: for each $t \in S^1$, define $g_t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by considering $\mathbb{R}^{2n}$ as the direct sum of n copies of $\mathbb{R}^2$ and rotating each $\mathbb{R}^2$ summand an angle t. Let X be the vector field on $\mathbb{R}^{2n}$ defined by $X(x) = \frac{dg_t(x)}{dt}\big|_{t=0}$ for any $x \in \mathbb{R}^{2n}$.

(a) Find the Lie derivative $\mathcal{L}_X\omega$ and a function f on $\mathbb{R}^{2n}$ such that $df = i_X\omega$.

(b) The S^1-action above induces an action on S^{2n-1}. Let $\mathbb{P}^{n-1}$ be the quotient space of S^{2n-1} by this S^1-action. Show that the quotient space $\mathbb{P}^{n-1}$ has a natural smooth structure and that the tangent space of $\mathbb{P}^{n-1}$ at any point $\underline{x}$ can be identified with the quotient of the tangent space T_xS^{2n-1} by the line spanned by $X(x)$, for any $x \in \underline{x}$. Here $\underline{x}$ is the orbit of x under the S^1-action.

(c) Show that ω descends to a well-defined 2-form on the quotient space $\mathbb{P}^{n-1}$ and that the 2-form so defined is closed.

(d) Is the closed form in (c) exact?

Hint: For (c) and (d), use (a) and (b).

6. Suppose that $f : S^n \to S^n$ is a smooth map of degree not equal to $(-1)^{n+1}$. Show that f has a fixed point.

7. (a) Let G be a finitely presented group. Show that there is a topological space X with fundamental group $\pi_1(X) \cong G$.

(b) Give an example of X in the case $G = \mathbb{Z} * \mathbb{Z}$, the free group on two generators.

(c) How many connected, 2-sheeted covering spaces does the space X from (b) have?

8. Let G be a connected topological group. Show that $\pi_1(G)$ is a commutative group.

9. Show that if $\mathbb{R}^m$ and $\mathbb{R}^n$ are homeomorphic, then $m = n$.

10. Let N_g be the nonorientable surface of genus g, that is, the connected sum of g copies of $\mathbb{RP}^2$. Calculate the fundamental group and homology groups of N_g.