QUALIFYING EXAM
Geometry/Topology
September 2011

Attempt all problems. Each problem is worth 10 points. Justify your answers carefully.

1. Let M be an (abstract) compact smooth manifold. Prove that there exists some $n \in \mathbb{Z}^+$ such that M can be smoothly embedded in the Euclidean space $\mathbb{R}^n$.

2. Prove that the real projective space $\mathbb{RP}^n$ is a smooth manifold of dimension n.

3. Let M be a compact, simply connected smooth manifold of dimension n. Prove that there is no smooth immersion $f : M \to T^n$, where $T^n = S^1 \times \cdots \times S^1$ is the n-torus.

4. Give a topological proof of the Fundamental Theorem of Algebra: any non-constant single-variable polynomial with complex coefficients has at least one complex root.

5. Let $f : M \to N$ be a smooth map between two manifolds M and N. Let α be a p-form on N. Show that $d(f^*\alpha) = f^*(d\alpha)$.

6. (a) What are the de Rham cohomology groups of a smooth manifold?
(b) State de Rham's theorem.

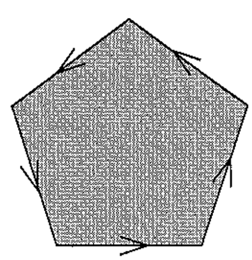
7. Consider the form

$$\omega = (x^2 + x + y)dy \wedge dz$$

on $\mathbb{R}^3$. Let $S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ be the unit sphere, and $i : S^2 \to \mathbb{R}^3$ the inclusion.
(a) Calculate $\int_{S^2} \omega$.
(b) Construct a closed form α on $\mathbb{R}^3$ such that $i^*\alpha = i^*\omega$, or show that such a form α does not exist.

8. (a) Let M be a Möbius band. Using homology, show that there is no retraction from M to ∂M.
(b) Let K be a Klein bottle. Show that there exist homotopically nontrivial simple closed curves γ_1 and γ_2 on K such that K retracts to γ_1, but does not retract to γ_2.

9. Let X be the topological space obtained from a pentagon by identifying its edges as in the picture:



Calculate the homology and cohomology groups of X with integer coefficients.

10. Let X, Y be topological spaces and $f, g : X \to Y$ two continuous maps. Consider the space Z obtained from the disjoint union $Y \amalg (X \times [0,1])$ by identifying $(x, 0) \sim f(x)$ and $(x, 1) \sim g(x)$ for all $x \in X$. Show that there is a long exact sequence of the form:

$$\ldots \longrightarrow H_n(X) \longrightarrow H_n(Y) \longrightarrow H_n(Z) \longrightarrow H_{n-1}(X) \longrightarrow \ldots$$