Spring 2011 Geometry/Topology Qualifying Exam

Do all questions. Each question counts the same amount.
Answers should be concise but complete.

1. Show that if V is a smooth vector field on a (smooth) manifold of dimension n and if V(p) is nonzero for some point of p, then there is a coordinate system defined in a neighborhood of p , say $(x_1, \ldots, x_n)$, such that on a neighborhood of p, V= the x_1 coordinate vector field.

2. (a) Demonstrate the formula $L_X = d\, i_X + i_X d$, where L is the Lie derivative and i is the interior product.
(b) Use this formula to show that a vector-field X on R^3 has a flow(locally) that preserves volume if and only if the divergence of X is everywhere 0.
[Here divergence is the classical operator that takes a vector field with components f, g ,h to the function $f_x + g_y + h_z$, , in the usual partial derivative notation f_x = x-partial of f, etc.]

3. (a) Explain some systematic reason why there is a closed 2-form on $R^3 - \{(0,0,0)\}$ (Euclidean 3-space with one point removed) that is not exact.
You may do this by exhibiting such a form explicitly and checking that it is closed but not exact or you may argue using theorems that such a form must exist.
(b) With φ such a form(as in part 1), discuss why , for any smooth map of S^2 to itself , the number

(the integral of $F^*\varphi$ over S^2)/ (the integral of φ over S^2)

is the degree of f.
[Note that this includes explaining why the denominator integral cannot be 0.].

4. Show without using deRham's Theorem (but you may use the Poincare Lemma without proof), that a 2-form φ on the 2-sphere S^2 that has integral 0 is exact, i.e., equal to d□ for some 1-form □ on S^2 .

5. Suppose that $V: U \to S^2$ is a smooth map, considered as a vector field of unit vectors, where
U= R^3 with a finite number points $p_1, \ldots, p_n$ removed ,
all these points lying strictly inside the unit sphere S^2.

Explain carefully, from basic facts about critical values and critical points and the like, why the degree of $V|S^2: S^2 \to S^2$ is equal to the sum of the indices of the vector field V at the points $p_1, \ldots, p_n$.

6. (a) Explain what a short exact sequence of chain complexes is.
(b) Describe how a short exact sequence of chain complexes gives rise to a long exact sequence in homology. Include how the connecting homomorphism(where the dimension changes) arises. You do not need to prove exactness of the sequence.

7. (a) Define complex projective space CP^n, $n = 1,2,3,\ldots$
(b) Compute the homology and cohomology of CP^n, Z coefficients.
(Any method is allowed. Cell complexes are particularly simple to use. Be sure to explain what the attaching maps are if you adopt this approach).

8. (a) Find the Z-coefficient homology of RP^2 by any systematic method.
(b) Explain explicitly (not using the Kunneth Theorem) how a nonzero element of the 3-homologywith Z-coefficients of $RP^2 \times RP^2$ arises.

9. (a) State the Lefschetz Fixed Point Theorem.
(b) Show that the Lefschetz number of any map from CP^{2n} to itself is nonzero and hence that every map from CP^{2n} to itself has a fixed point(Suggestion: The action of the map on cohomology with Z coefficients is determined by what happens to the 2nd cohomology since the whole cohomology ring is generated by the 2nd cohomology).

10. Compute explicitly the simplicial homology, Z coefficients, of the surface of a tetrahedron, thus obtaining the homology of the 2-sphere.