Qualifying Exam: Geometry/Topology Fall 2012

Instructions: Do all 10 problems.

Problem 1: (a) Show that the Lie group $SL_2(\mathbb{R}) = \{A \in M_{2\times 2}(\mathbb{R}) \mid \det(A) = 1\}$ is diffeomorphic to $S^1 \times \mathbb{R}^2$.

(b) Show that the Lie group $SL_2(\mathbb{C}) = \{A \in M_{2\times 2}(\mathbb{C}) \mid \det(A) = 1\}$ is diffeomorphic to $S^3 \times \mathbb{R}^3$.

Problem 2: For $n \geq 1$, construct an everywhere non-vanishing smooth vector field on the odd-dimensional real projective space $\mathbb{RP}^{2n-1}$.

Problem 3: Let $M^m \subset \mathbb{R}^n$ be a smooth submanifold of dimension $m < n-2$. Show that its complement $\mathbb{R}^n \setminus M$ is connected and simply connected.

Problem 4: (a) Show that for any $n \geq 1$ and $k \in \mathbb{Z}$, there exists a continuous map $f : S^n \to S^n$ of degree k.

(b) Let X be a compact, oriented n-dimensional manifold. Show that for any $k \in \mathbb{Z}$, there exists a continuous map $f : X \to S^n$ of degree k.

Problem 5: Assume that $\Delta = \{X_1, ..., X_k\}$ is a k-dimensional distribution spanned by vector fields on an open set $\Omega \subset M^n$ in an n-dimensional manifold. For each open subset $V \subset \Omega$ define

$$\mathcal{Z}_V = \{u \in C^\infty(V) \mid X_1 u = 0, ..., X_k u = 0\}$$

Show that the following two statements are equivalent:

(a) The distribution Δ is integrable.

(b) For each $x \in \Omega$ there exists an open neighborhood $x \in V \subset \Omega$ and $n-k$ functions $u_1, ..., u_{n-k} \in \mathcal{Z}_V$ such that the differentials $du_1, ..., du_{n-k}$ are linearly independent at each point in V.

Problem 6: On $\mathbb{R}^n - \{0\}$ define the $(n-1)$-forms

$$\sigma = \sum_{i=1}^{n} (-1)^{i-1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

$$\omega = \frac{1}{|x|^n} \sum_{i=1}^{n} (-1)^{i-1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

(a) Show that $\omega = r^* \circ i^*(\sigma)$, where $i : S^{n-1} \to \mathbb{R}^n - \{0\}$ is the natural inclusion of the unit sphere and $r(x) = \frac{x}{|x|} : \mathbb{R}^n - \{0\} \to S^{n-1}$ the natural retraction.

(b) Show that σ is not a closed form.
(c) Show that ω is a closed form that is not exact.

Problem 7: Let $n \geq 0$ be an integer. Let M be a compact, orientable, smooth manifold of dimension $4n+2$. Show that $\dim H^{2n+1}(M; \mathbb{R})$ is even.

Problem 8: Show that there is no compact three-dimensional manifold M whose boundary is the real projective space $\mathbb{RP}^2$.

Problem 9: Consider the coordinate axes in $\mathbb{R}^n$:

$$L_i = \{(x_1, \ldots, x_n) \mid x_j = 0 \text{ for all } j \neq i\}$$

Calculate the homology groups of the complement $\mathbb{R}^n \setminus (L_1 \cup \ldots \cup L_n)$.

Problem 10: (a) Let X be a finite CW complex. Explain how the homology groups of X are related to the homology groups of $X \times S^1$.

(b) For each integer $n \geq 0$, give an example of a compact smooth manifold of dimension $2n+1$ such that $H_i(X) = \mathbb{Z}$ for all $i = 0, \ldots, 2n+1$.