Qualifying Examination: Geometry/Topology March 27,2012
Do all problems. All problems count equally.

1 Explain in detail from the viewpoint of transversality theory, why the sum of the indices of a vector field with isolated zeroes on a compact orientable manifold M is independent of what vector field we choose.

2 Call the index sum in problem 1 the Euler characteristic $\chi(M)$. Explain why the Euler characteristic of a genus g surface (2-sphere with g handles attached) is 2-2g. [Do this explicitly: do NOT appeal to the theorem that the Euler characteristic in the vector field sense indicated is computable from homological information. That comes next!]

3 Suppose that M is a triangulated compact orientable manifold, i.e., a manifold M represented as a finite simplicial complex.
(a) Show that the alternating sum of the Betti numbers $b_0 - b_1 + b_2 - \ldots$ (where b_k= rank of the kth homology group with real coefficients) is equal to the alternating sum
(number of vertices) - (number of faces) + (number of 2-simplices) -…

(b) Show that there is a vector field with the sum of its indices equal to the number described in part (a).
[You do not need to worry about smoothness of the vector field –just describe how to build it. In part (a), the result should follow from some dimension counting.]

4 Suppose V is a smooth (C-infinity) vector field on R^3 that is nonzero at (0,0,0). The vector field is said to be gradient-like at (0,0,0) if there is a neighborhood of (0,0,0) and a nowhere zero smooth function $\lambda(x,y,z)$ on that neighborhood such that λ V is the gradient of some smooth function in some (possibly smaller) neighborhood of (0,0,0).
(a) Write V=(P,Q,R). Show by example that there are functions P,Q, R for which V is not gradient-like in a neighborhood of (0,0,0).
[Suggestion: the orthogonal complement of V taken at each point would have to be an integrable 2-plane field]
(b) Derive a general differential condition on (P,Q,R) which is necessary and sufficient for V to be gradient-like in a neighborhood of (0,0,0)

5 (a) Define carefully the “boundary map” which defines the H_n to H_{n-1} mapping that arises in the long exact sequence arising from a short exact sequence of chain complexes.
(b) Prove that the kernel of the boundary map is equal to the image of the map into the H_n.

6 Compute the homology of the real projective space RP^n for each $n>1$.

7 (a) Define complex projective space CP^n (n=1,2,3,…)
(b) Show that CP^n is compact for all n.
(c) Show that CP^n has a cell decomposition with one cell in each dimension 0, 2, 4,…,2n and no other cells. Include a careful description of the attaching maps.

8 Suppose a compact (real) manifold M has a (finite) cell decomposition with only even dimensional cells. Is M necessarily orientable? Justify your answer.

9 Suppose that a finite group Γ acts smoothly on a compact manifold M and that the action is free, i.e. $\gamma(x) = x$ for some x in M if and only if γ= the identity of the group Γ.
(a) Show that M/Γ is a manifold (i.e., can be made a manifold in a natural way)
(b) Show that $M \to M/\Gamma$ is a covering space.
(c) If the kth deRham cohomology of M is 0, some particular $k>0$, then is the kth deRham cohomology of M/Γ necessarily 0? Prove your answer.

10 Let $M= RP^2 \times RP^2$ (where RP^2 is real projective 2-space). In a product manifold like that, homology elements can arise by taking in effect the product of a cycle in one factor with a cycle in the other factor.
Show that in the case of this particular M, there is an element in the 3-homology with Z coefficients that does not arise in this way by exhibiting such an element explicitly, e.g. in terms of a cell decomposition.