QUALIFYING EXAM

Geometry/Topology

September 2013

Answer all 10 questions. Each problem is worth 10 points. Justify your answers carefully.

1. Let $f : M \to N$ be a nonsingular smooth map between connected manifolds of the same dimension. Answer the following questions with a proof or counter-example.

(a) Is f necessarily injective or surjective?

(b) Is f necessarily a covering map when N is compact?

(c) Is f necessarily an open map?

(d) Is f necessarily a closed map?

2. Let M be a connected compact manifold with non-empty boundary ∂M. Show that M does not retract onto ∂M.

3. Let $M, N \subset \mathbb{R}^{p+1}$ be two compact, smooth, oriented submanifolds of dimensions m and n, respectively, such that $m + n = p$. Suppose that $M \cap N = \emptyset$. Consider the linking map

$$\lambda : M \times N \to S^p, \quad \lambda(x, y) = \frac{x - y}{\|x - y\|}.$$

The degree of λ is called the linking number $l(M, N)$.

(a) Show that $l(M, N) = (-1)^{(m+1)(n+1)} l(N, M)$.

(b) Show that if M is the boundary of an oriented submanifold $W \subset \mathbb{R}^{p+1}$ disjoint from N, then $l(M, N) = 0$.

4. Let ω be a 1-form on a connected manifold M. Show that ω is exact, i.e., $\omega = df$ for some function f, if and only if for all piecewise smooth closed curves $c : S^1 \to M$ it follows that $\int_c \omega = 0$.

5. Let ω be a smooth, nowhere vanishing 1-form on a three-dimensional smooth manifold M^3.

(a) Show that $\ker \omega$ is an integrable distribution on M if and only if $\omega \wedge d\omega = 0$.

(b) Give an example of a codimension one distribution on $\mathbb{R}^3$ that is not integrable.

6. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function.

(a) Define the gradient ∇f as a vector field dual to the differential df.

(b) Define the Hessian $\text{Hess} f\,(X,Y)$ as a symmetric $(0,2)$- tensor.

(c) If the usual Euclidean inner product between tangent vectors in $T_p\mathbb{R}^n$ is denoted $g\,(X,Y) = X \cdot Y$ show that

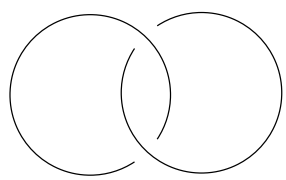
$$\text{Hess} f\,(X,Y) = \frac{1}{2}\left(\mathscr{L}_{\nabla f}\, g\right)(X,Y)$$

Here $\mathscr{L}_Z g$ is the Lie derivative of g in the direction of Z.

7. Let $M = T^2 - D^2$ be the complement of a disk inside the two-torus. Determine all connected surfaces that can be described as 3-fold covers of M.

8. Let $n > 0$ be an integer and let A be an abelian group with a finite presentation by generators and relations. Show that there exists a topological space X with $H_n(X) \cong A$.

9. Let $H \subset S^3$ be the Hopf link, shown in the figure



Compute the fundamental group and the homology groups of the complement $S^3 - H$.

10. Let $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ be the group of quaternions, with relations $i^2 = j^2 = -1$, $ij = -ji = k$. The multiplicative group $\mathbb{H}^* = \mathbb{H} - \{0\}$ acts on $\mathbb{H}^n - \{0\}$ by left multiplication. The quotient $\mathbb{HP}^{n-1} = (\mathbb{H}^n - \{0\})/\mathbb{H}^*$ is called the quaternionic projective space. Calculate its homology groups.