Qualifying Exam
Geometry and Topology, March 23, 2013

1. Let $\mathrm{Mat}_{m\times n}(\mathbb{R})$ be the space of $m \times n$ matrices with real valued entries.

   (a) Show that the subset $S \subset \mathrm{Mat}_{m\times n}(\mathbb{R})$ of rank 1 matrices form a submanifold of dimension $m+n-1$.

   (b) Show that the subset $T \subset \mathrm{Mat}_{m\times n}(\mathbb{R})$ of rank k matrices form a submanifold of dimension $k(m+n-k)$.

2. Let M be a smooth manifold and $\omega \in \Omega^1(M)$ a smooth 1-form.

   (a) Define the line integral
   $$\int_c \omega$$
   along piecewise smooth curves $c : [0,1] \to M$.

   (b) Show that $\omega = df$ for a smooth function $f : M \to \mathbb{R}$ if and only if $\int_c \omega = 0$ for all closed curves $c : [0,1] \to M$, i.e., $c(0) = c(1)$.

3. Let $S_1, S_2 \subset M$ be smooth embedded submanifolds.

   (a) Define what it means for S_1, S_2 to be transversal.

   (b) Show that if $S_1, S_2 \subset M$ are transversal then $S_1 \cap S_2 \subset M$ is a smooth embedded submanifold of dimension $\dim S_1 + \dim S_2 - \dim M$.

4. Let $S \subset M$ be given as $F^{-1}(c)$ where $F = \left(F^1, ..., F^k\right) : M \to \mathbb{R}^k$ is smooth and $c \in \mathbb{R}^k$ is a regular value for F. If $f : M \to \mathbb{R}$ is smooth, show that its restriction $f|_C$ to a submanifold $C \subset M$ has a critical point at $p \in C$ if and only if there exist constants $\lambda_1, ..., \lambda_k$ such that
   $$df_p = \sum \lambda_i dF^i_p$$
   where $dg_p : T_pM \to \mathbb{R}$ denotes the differential at p of a smooth function g.

5. Let M be a smooth, orientable, compact manifold with boundary ∂M. Show that there is no (smooth) retract $r : M \to \partial M$.

6. Let $A \in \mathrm{Gl}_{n+1}(\mathbb{C})$.

   (a) Show that A defines a smooth map $A : \mathbb{CP}^n \to \mathbb{CP}^n$.

(b) Show that the fixed points of $A : \mathbb{CP}^n \to \mathbb{CP}^n$ correspond to eigenvectors for the original matrix.

(c) Show that $A : \mathbb{CP}^n \to \mathbb{CP}^n$ is a Lefschetz map if the eigenvalues of A all have multiplicity 1.

(d) Show that the Lefschetz number of $A : \mathbb{CP}^n \to \mathbb{CP}^n$ is $n+1$. Hint: You are allowed to use that $\mathrm{Gl}_{n+1}(\mathbb{C})$ is connected.

7. Let $F : S^n \to S^n$ be a continuous map.

(a) Define the degree $\deg F$ of F and show that when F is smooth

$$\deg F \int_{S^n} \omega = \int_{S^n} F^* \omega$$

for all $\omega \in \Omega^n(S^n)$.

(b) Show that if F has no fixed points then $\deg F = (-1)^{n+1}$.

8. Let $f : S^{n-1} \to S^{n-1}$ be a continuous map and D^n the disk with $\partial D^n = S^{n-1}$.

(a) Define the adjunction space $D^n \cup_f D^n$.

(b) Let $\deg f = k$ and compute the homology groups $H_p\left(D^n \cup_f D^n, \mathbb{Z}\right)$ for $p = 0, 1, \ldots$.

(c) Assume that f is a homeomorphism, show that $D^n \cup_f D^n$ is homeomorphic to S^n.

9. Let $F : M \to N$ be a finite covering map between closed manifolds. Either prove or find counter examples to the following questions.

(a) Do M and N have the same fundamental groups?

(b) Do M and N have the same de Rham cohomology groups?

(c) When M is simply connected, do M and N have the same singular homology groups?

10. Let $A \subset X$ be a subspace of a topological space. Define the relative singular homology groups $H_p(X, A)$ and show that there is a long exact sequence

$$\to H_p(A) \to H_p(X) \to H_p(X, A) \to H_{p-1}(A) \to$$