QUALIFYING EXAM

Geometry/Topology

September 2014

Answer all 10 questions. Each problem is worth 10 points. Justify your answers carefully.

1. Let $f : M \to N$ be a proper immersion between connected manifolds of the same dimension. Show that f is a covering map.

2. Let $M^m \subset \mathbb{R}^n$ be a closed connected submanifold of dimension m.

(a) Show that $\mathbb{R}^n \backslash M^m$ is connected when $m \leq n-2$.

(b) When $m = n-1$ show that $\mathbb{R}^n \backslash M^m$ is disconnected by showing that the mod 2 intersection number $I_2(f, M) = 0$ for all smooth maps $f : S^1 \to \mathbb{R}^n$.

3. Let ω be an n-form on a closed connected non-orientable n-manifold M and $\pi : \mathcal{O} \to M$ the orientation cover.

(a) Show that $\pi^*\omega$ is exact.

(b) Show that ω is exact.

4. Show that for $n \geq 1$ any smooth map $f : S^{n-1} \to S^{n-1}$ has a smooth extension $F : D^n \to D^n$, where $D^n = \{x \in \mathbb{R} \mid |x| \leq 1\}$.

5. Let M be a smooth manifold and ω a nowhere vanishing 1-form on M. Show that ω is locally proportional to the differential of a function (i.e., around every point $p \in M$ there is a neighborhood $U \ni p$ and functions $f, \lambda : U \to \mathbb{R}$ such that $\omega = \lambda df$) if and only if $\omega \wedge d\omega = 0$.

6. Recall that the *rank* of a matrix is the dimension of the span of its row vectors. Show that the space of all 2×3 matrices of rank 1 forms a smooth manifold.

7. A compact surface of genus g, smoothly embedded in $\mathbb{R}^3$, bounds a compact region called a *handlebody* H.

(a) Prove that two copies of H glued together along their boundaries by the identity map produces a closed topological 3-manifold M.

(b) Compute the homology of M.

(c) Compute the relative homology of (M, H), where H is one of the two copies.

8. Consider the space $X = M_1 \cup M_2$, where M_1 and M_2 are Möbius bands and $M_1 \cap M_2 = \partial M_1 = \partial M_2$. Here a *Möbius band* is the quotient space $([-1,1]\times[-1,1])/((1,y) \sim (-1,-y))$.

(a) Determine the fundamental group of X.

(b) Is X homotopy equivalent to a compact orientable surface of genus g for some g?

9. Determine all the connected covering spaces of the wedge sum $\mathbb{RP}^{14} \vee \mathbb{RP}^{15}$.

10. Let D be the unit disk in the complex plane and S^1 be the unit circle in the complex plane. Consider the 2-dimensional torus $T^2 = S^1 \times S^1$ and two copies D_1 and D_2 of D. Let X be the quotient of the disjoint union $T^2 \sqcup D_1 \sqcup D_2$ by the equivalence relations

$$e^{i\theta} \sim (e^{ip\theta}, 1), \quad e^{i\phi} \sim (1, e^{iq\phi}),$$

where $e^{i\theta} \in D_1$, $e^{i\phi} \in D_2$, $(e^{ip\theta}, 1), (1, e^{iq\phi}) \in T^2$, and p, q are integers > 1. Compute the homology groups of X.