QUALIFYING EXAM

Geometry/Topology

September 2015

Answer all 10 questions. Each problem is worth 10 points. Justify your answers carefully.

1. Let $M_n(\mathbb{R})$ be the space of $n \times n$ matrices with real coefficients.

(a) Show that $SL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) = 1\}$ is a smooth submanifold of $M_n(\mathbb{R})$.

(b) Show that $SL(n, \mathbb{R})$ has trivial Euler characteristic.

2. Let $f, g : M \to N$ be smooth maps between smooth manifolds that are smoothly homotopic. Prove that if ω is a closed form on N, then $f^*\omega$ and $g^*\omega$ are cohomologous.

3. For two smooth vector fields X, Y on a smooth manifold M, prove the formula

$$[\mathcal{L}_X, i_Y]\omega = i_{[X,Y]}\omega,$$

where $\mathcal{L}_X$ is the Lie derivative in the direction of X, i_X is the interior product of X, and ω is a k-form for $k \geq 1$.

4. Let $M = \mathbb{R}^3/\mathbb{Z}^3$ be a three-dimensional torus and $C = \pi(L)$, where $L \subset \mathbb{R}^3$ is the oriented line segment from $(0, 1, 1)$ to $(1, 3, 5)$ and $\pi : \mathbb{R}^3 \to M$ is the quotient map. Find a differential form on M which represents the Poincaré dual of C.

5. Recall that the Hopf fibration $\pi : S^3 \to S^2$ is defined as follows: if we identify

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$$

and $S^2 = \mathbb{CP}^1$ with homogenous coordinates $[z_1, z_2]$, then $\pi(z_1, z_2) = [z_1, z_2]$. Show that π does not admit a section, i.e., a smooth map $s : S^2 \to S^3$ such that $\pi \circ s = id_{S^2}$.

6. Let $M^m \subset \mathbb{R}^n$ be a smooth submanifold of dimension $m < n - 2$. Show that its complement $\mathbb{R}^n - M$ is connected and simply-connected.

7. Show that there exists no smooth degree one map from $S^2 \times S^2$ to $\mathbb{CP}^2$.

8. Show that $\mathbb{CP}^{2n}$, $n \in \mathbb{Z}^+$, is not a covering space of any manifold except itself.

9. Given a continuous map $f : X \to Y$ between topological spaces, define

$$C_f = \left((X \times [0, 1]) \coprod Y\right) / \sim,$$

where $(x, 1) \sim f(x)$ for all $x \in X$ and $(x, 0) \sim (x', 0)$ for all $x, x' \in X$. Here $\coprod$ is the disjoint union. Show that there is a long exact sequence

$$\cdots \to H_{i+1}(X) \xrightarrow{f_*} H_{i+1}(Y) \to \widetilde{H}_{i+1}(C_f) \to H_i(X) \xrightarrow{f_*} H_i(Y) \to \cdots,$$

where f_* is the map on homology induced from f and $\widetilde{H}_i$ denotes the ith reduced homology group.

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10. Let $\mathbb{RP}^n$ be the real projective space given by $S^n/\sim$, where $S^n = \{\|x\| = 1\} \subset \mathbb{R}^{n+1}$ and $x \sim -x$ for all $x \in S^n$.

(a) Give a cell (CW) decomposition of $\mathbb{RP}^n$ for $n \geq 1$.

(b) Use the cell decomposition to compute the homology groups $H_k(\mathbb{RP}^n)$, $k \geq 0$.

(c) For which values of $n \geq 1$ is $\mathbb{RP}^n$ orientable? Explain.