QUALIFYING EXAM

Geometry/Topology

March 2015

Answer all 10 questions. Each problem is worth 10 points. Justify your answers carefully.

1. Let $M(n,m,k) \subset M(n,m)$ denote the space of $n \times m$-matrices of rank k. Show that $M(n,m,k)$ is a smooth manifold of dimension $nm - (n-k)(m-k)$.

2. Assume that $N \subset M$ is a codimension 1 properly embedded submanifold. Show that N can be written as $f^{-1}(0)$, where 0 is a regular value of a smooth function $f : M \to \mathbb{R}$, if and only if there is a vector field X on M that is transverse to N.

3. Consider two collections of 1-forms $\omega_1, ..., \omega_k$ and $\phi_1, ..., \phi_k$ on an n-dimensional manifold M. Assume that

$$\omega_1 \wedge \cdots \wedge \omega_k = \phi_1 \wedge \cdots \wedge \phi_k$$

never vanishes on M. Show that there are smooth functions $f_{ij} : M \to \mathbb{R}$ such that

$$\omega_i = \sum_{j=1}^{k} f_{ij}\phi_j,\ i = 1, ..., k.$$

4. Consider a smooth map $F : \mathbb{RP}^n \to \mathbb{RP}^n$.

(a) When n is even show that F has a fixed point.

(b) When n is odd give an example where F does not have a fixed point.

5. Assume we have a codimension 1 distribution $\Delta \subset TM$.

(a) Show if the quotient bundle TM/Δ is trivial (or equivalently that there is a vector field on M that never lies in Δ), then there is a 1-from ω on M such that $\Delta = \ker \omega$ everywhere on M.

(b) Give an example where TM/Δ is not trivial.

(c) With ω_1 and ω_2 as in (a) show that $\omega_1 \wedge d\omega_1 = f^2 \omega_2 \wedge d\omega_2$ for a smooth function f that never vanishes.

(d) If ω is as in (a) and $\omega \wedge d\omega \neq 0$, show that Δ is not integrable.

6. Let

$$\omega = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}$$

be a 2-form defined on $\mathbb{R}^3 - \{0\}$. If $i : S^2 = \{x^2 + y^2 + z^2 = 1\} \to \mathbb{R}^3$ is the inclusion, then compute $\int_{S^2} i^*\omega$. Also compute $\int_{S^2} j^*\omega$, where $j : S^2 \to \mathbb{R}^3$ maps $(x, y, z) \to (3x, 2y, 8z)$.

7. Define the de Rham cohomology groups $H^i_{dR}(M)$ of a manifold M and compute $H^i_{dR}(S^1)$, $S^1 = \mathbb{R}/\mathbb{Z}$, $i = 0, 1, \ldots$, directly from the definition.

8. Let X be a CW complex consisting one vertex p, 2 edges a and b, and two 2-cells f_1 and f_2, where the boundaries of a and b map to p, the boundary of f_1 is mapped to the loop ab^2 (that is first a and then b twice), and the boundary of f_2 is mapped to the loop ba^2.

(a) Compute the fundamental group $\pi_1(X)$ of X. Is it a finite group?

(b) Compute the homology groups $H_i(X)$, $i = 0, 1, \ldots$, of X.

9. Let X, Y be topological spaces and let $f, g : X \to Y$ be two continuous maps. Consider the space Z obtained from the disjoint union $(X \times [0,1]) \sqcup Y$ by identifying $(x, 0) \sim f(x)$ and $(x, 1) \sim g(x)$ for all $x \in X$. Show that there is a long exact sequence of the form:

$$\cdots \to H_i(X) \overset{a}{\to} H_i(Y) \overset{b}{\to} H_i(Z) \overset{c}{\to} H_{i-1}(X) \to \ldots$$

Also describe the maps a, b, c.

10. Let $n \geq 0$ be an integer. Let M be a compact, orientable, smooth manifold of dimension $4n + 2$. Show that $\dim H^{2n+1}(M; \mathbb{R})$ is even.