QUALIFYING EXAM
Geometry/Topology
September 2016

Attempt all ten problems. Each problem is worth 10 points. Justify your answers carefully.

1. Let M be a smooth manifold. Prove that for any two disjoint closed subsets $A, B \subset M$ there is a smooth function $f : M \to \mathbb{R}$ such that $f = 0$ on A and $f = 1$ on B.

2. Let $M \subset \mathbb{R}^N$ be a smooth k-dimensional submanifold. Prove that M can be immersed into $\mathbb{R}^{2k}$.

3. Let $U_1, \dots, U_n$ be n bounded, connected, open subsets of $\mathbb{R}^n$. Prove that there exists an $(n-1)$-dimensional hyperplane $H \subset \mathbb{R}^N$ that bisects every U_i; i.e., if A and B are the two half-spaces that form $\mathbb{R}^n \setminus H$, then

$$\text{volume}(U_i \cap A) = \text{volume}(U_i \cap B)$$

for all $i = 1, \dots, n$.

4. Show that

$$D = \ker(dx_3 - x_1 dx_2) \cap \ker(dx_1 - x_4 dx_2) \subset T\mathbb{R}^4$$

is a smooth distribution of rank two, and determine whether D is integrable.

5. (a) Let M be a smooth compact manifold and $N \subset M$ a smooth compact submanifold. Explain (in terms of integrals, without reference to cohomology) what it means for a closed differential form ω to be Poincaré dual to N.

In parts (b) and (c), you are free to use your knowledge of homology and cohomology:

(b) Let $M = T^2$ be the two-dimensional torus with coordinates $(x, y) \in (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z}) \cong T^2$. Identify a submanifold $N \subset M$ Poincaré dual to the form dy, and show that they are indeed dual.

(c) Give an example of a closed 1-form on T^2 that is not Poincaré dual to any submanifold.

6. Let M be a smooth, compact, oriented n-dimensional manifold. Suppose that the Euler characteristic of M is zero.

(a) Show that M admits a nowhere vanishing vector field.

(b) A *Lorentzian metric* on M is a smoothly varying, symmetric bilinear form

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

of signature $(n-1, 1)$; that is, at every $p \in M$ there exists a basis $e_1, \dots, e_n$ of T_pM such that, with respect to this basis, g_p is a diagonal matrix with $n-1$ entries of 1 and one entry of -1. Prove that M admits a Lorentzian metric.

7. Let X be a connected CW-complex with $\pi_1(X, x)$ finite. Show that any map $X \to (S^1)^n$ is null-homotopic.

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8. Consider $X = \mathbb{RP}^2 \vee \mathbb{RP}^2$. Let a generate π_1 of the first summand and b generate π_1 of the second summand. For $n \geq 1$, describe the covering space $p : Y \to X$ such that $p_*(\pi_1(Y))$ is the subgroup of $\pi_1(X)$ generated by $(ab)^n$. (A drawing and a short explanation would suffice.)

9. Let $S^2 \xleftarrow{q_1} S^2 \vee S^2 \xrightarrow{q_2} S^2$ be the maps that crush out one of the two summands. Let $f\colon S^2 \to S^2 \vee S^2$ be a map such that $q_i \circ f\colon S^2 \to S^2$ is a map of degree d_i. Compute the homology groups of $(S^2 \vee S^2) \cup_f D^3$.

10. If $f\colon X \to X$ is a self-map, then the "mapping torus of f" is the quotient

$$T_f := \big(X \times [0,1]\big)/(x,0) \sim (f(x),1), \forall x \in X.$$

For $n \in \mathbb{Z}$, let f_n be a degree n map on S^3. Compute the homology groups of T_{f_n}.