

# QUALIFYING EXAM

## Geometry/Topology

March 2016

Answer all 10 questions. Each problem is worth 10 points. Justify all your steps.

1. Consider the space of all straight lines in  $\mathbb{R}^2$  (not necessarily those passing through the origin). Explain how to give it the structure of a smooth manifold. Is it orientable?
2. Let  $X$  and  $Y$  be submanifolds of  $\mathbb{R}^n$ . Prove that, for almost every  $a \in \mathbb{R}^n$ , the translate  $X + a$  intersects  $Y$  transversely.
3. Consider the vector field  $X(z) = z^{2016} + 2016z^{2015} + 2016$  on  $\mathbb{C} = \mathbb{R}^2$ . (By this we mean the following: take a complex coordinate  $z$  on  $\mathbb{C}$ , identify  $T_z\mathbb{C} = \mathbb{C}$ , and write  $X(z) = z^{2016} + 2016z^{2015} + 2016 \in T_z\mathbb{C}$ .) Compute the sum of the indices of  $X$  over all the zeros of  $X$ .
4. Let  $M$  be a compact odd-dimensional manifold with nonempty boundary  $\partial M$ . Show that the Euler characteristics of  $M$  and  $\partial M$  are related by:

$$\chi(M) = \frac{1}{2}\chi(\partial M).$$

5. Let  $M$  be a compact oriented  $n$ -manifold with de Rham cohomology group  $H_{dR}^1(M; \mathbb{R}) = 0$  and let  $T^n$  be the  $n$ -dimensional torus. For which integers  $k$  does there exist a smooth map  $f: M \rightarrow T^n$  of degree  $k$ ?
6. Let  $T^2 = \mathbb{R}^2/\mathbb{Z}^2$  be the two-dimensional torus with coordinates  $(x, y) \in \mathbb{R}^2$  and let  $p \in T^2$ .
  - (a) Compute the de Rham cohomology of the punctured torus  $T^2 - \{p\}$ .
  - (b) Is the volume form  $\omega = dx \wedge dy$  exact on  $T^2 - \{p\}$ ?

7. Exhibit a space whose fundamental group is isomorphic to  $(\mathbb{Z}/m\mathbb{Z}) * (\mathbb{Z}/n\mathbb{Z})$ , where  $\mathbb{Z}/k\mathbb{Z}$  denotes the integers modulo  $k$  and  $*$  denotes the free product. Exhibit another space whose fundamental group is isomorphic to  $(\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})$ .
8. Let  $L_x$  be the  $x$ -axis,  $L_y$  be the  $y$ -axis, and  $L_z$  be the  $z$ -axis of  $\mathbb{R}^3$ . Compute

$$\pi_1(\mathbb{R}^3 - L_x - L_y - L_z).$$

9. Let  $X$  be a topological space and  $p \in X$ . The *reduced suspension*  $\Sigma X$  of  $X$  is the space obtained from  $X \times [0, 1]$  by contracting  $(X \times \{0, 1\}) \cup (\{p\} \times [0, 1])$  to a point. Describe the relation between the homology groups of  $X$  and  $\Sigma X$ .

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10. Consider the 3-form on  $\mathbb{R}^4$  given by

$$\alpha = x_1 dx_2 \wedge dx_3 \wedge dx_4 - x_2 dx_1 \wedge dx_3 \wedge dx_4 + x_3 dx_1 \wedge dx_2 \wedge dx_4 - x_4 dx_1 \wedge dx_2 \wedge dx_3.$$

Let  $S^3$  be the unit sphere in  $\mathbb{R}^4$  and let  $\iota : S^3 \rightarrow \mathbb{R}^4$  be the inclusion map.

(a) Evaluate  $\int_{S^3} \iota^* \alpha$ .

(b) Let  $\gamma$  be the 3-form on  $\mathbb{R}^4 - \{0\}$  given by:

$$\gamma = \frac{\alpha}{(x_1^2 + x_2^2 + x_3^2 + x_4^2)^k}$$

for  $k \in \mathbb{R}$ . Determine the values of  $k$  for which  $\gamma$  is closed and those for which it is exact.