Attempt all ten problems. Each problem is worth 10 points. Justify your answers carefully.

1. Let M be a smooth manifold. Verify the following identity for vector fields X, Y and a 1-form ω on M:

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]).$$

2. Let $M_n(\mathbb{R})$ be the space of all $n \times n$ matrices with real coefficients.

   (i) (5 pts) Show that $O(n) = \{A \in M_n(\mathbb{R}) \mid AA^T = \mathrm{id}\}$ is a smooth submanifold of $M_n(\mathbb{R})$. Here A^T is the transpose of A.

   (ii) (5 pts) Show that $O(n)$ has trivial tangent bundle.

3. The Hopf fibration $\pi : S^3 \to S^2$ is defined as follows: if we identify

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\},$$

   and $S^2 = \mathbb{CP}^1$ with homogeneous coordinates $[z_1, z_2]$, then $\pi(z_1, z_2) = [z_1, z_2]$. There is another fibration $p : \mathbb{S}TS^2 \to S^2$, called the *unit tangent bundle*, whose fiber over $x \in S^2$ consists of the tangent vectors in $T_x S^2$ of unit length (here we may measure the length of a tangent vector by viewing S^2 as a submanifold of $\mathbb{R}^3$). Show that there is a covering map $f : S^3 \to \mathbb{S}TS^2$ of degree 2 satisfying $p \circ f = \pi$.

4. Consider the differential 1-form $\omega = x\,dy - y\,dx + dz$ in $\mathbb{R}^3$ with coordinates (x, y, z). Prove that $f\omega$ is not closed for any nowhere zero function $f : \mathbb{R}^3 \to \mathbb{R}$.

5. Let x, y, z denote the standard Euclidean coordinates on $\mathbb{R}^3$ and let dA denote the standard area form on $S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$. Determine the values of $n = 0, 1, 2, \ldots$ for which $\omega = z^n dA$ is an exact 2-form on S^2.

6. (i) (3 pts) Define what it means for a manifold M to be orientable. (You can give any one of the many equivalent definitions.)

   (ii) (7 pts) Show that every nonorientable connected manifold M admits a connected, oriented double cover.

7. Let M be a smooth, compact, connected, oriented n-dimensional manifold (without boundary).

   (i) (5 pts) Show that if the Euler characteristic of M is zero, then M admits a nowhere vanishing vector field.

   (ii) (5 pts) If M is a surface of genus g, then what is $\min_v(\#$ zeros of $v)$, where v ranges over vector fields on M whose zeros are isolated and have index ± 1? Give a proof.

8. Let $M = [0,1] \times [0,1]/ \sim_0$, where $(x,1) \sim_0 (1-x,0)$ for all $x \in [0,1]$, and let $X = (M \times \{0,1\})/ \sim_1$, where $(y,1) \sim_1 (y,0)$ for all $y \in \partial M$. Determine the fundamental group of X.

9. A compact surface (without boundary) of genus g, embedded in $\mathbb{R}^3$ in the standard way (see below for the case $g = 3$), bounds a compact 3-dimensional region called a *handlebody* H (the region "inside" the surface in the following figure).
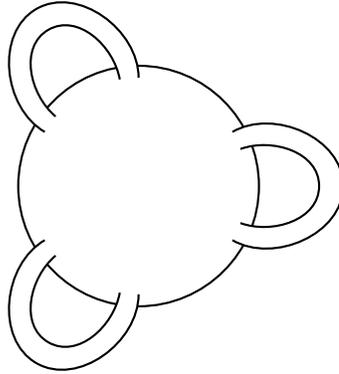
Let $X = (H \times \{0,1,2\})/ \sim$, where $(x,i) \sim (x,j)$ for all $x \in \partial H$ and $i, j \in \{0,1,2\}$. Compute the homology of X.

10. (i) (3 pts) Let A be a single circle in $\mathbb{R}^3$. Compute $\pi_1(\mathbb{R}^3 - A, *)$.

(ii) (3 pts) Let A and B be disjoint circles in $\mathbb{R}^3$, supported in the upper and lower half space, respectively. Compute $\pi_1(\mathbb{R}^3 - (A \cup B), *)$.

(iii) (4 pts) How does $\pi_1(\mathbb{R}^3 - (A \cup B), *)$ change if the circles are linked as below?