Attempt all ten problems. Each problem is worth 10 points. Justify your answers carefully.

1. Let M be a connected smooth manifold of dimension at least two. Prove that for any $2n$ distinct points

$$x_1, \ldots, x_n, y_1, \ldots, y_n \in M$$

there exists a diffeomorphism $f : M \to M$ such that $f(x_i) = y_i$ for $i = 1, \ldots, n$.

2. Let $M_{2n \times 2n}(\mathbb{R}) = \mathbb{R}^{4n^2}$ be the space of $2n \times 2n$ real matrices. Consider the following matrix in block form

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in M_{2n \times 2n}(\mathbb{R}),$$

where I_n is the $n \times n$ identity matrix. Show that the subspace

$$S = \{A \in M_{2n \times 2n}(\mathbb{R}) \mid A^T \Omega A = \Omega\}$$

is a smooth submanifold of $M_{2n \times 2n}(\mathbb{R})$, and compute its dimension. (Here A^T denotes the transpose of A.)

3. Use the Poincaré-Hopf index theorem to calculate the Euler characteristic of the n-dimensional sphere S^n. (You must compute the indices in local coordinates. Drawings do not suffice!)

4. (a) State the Cartan formula (also known as Cartan's magic formula) for the Lie derivative of a differential form with respect to a vector field.

(b) Use this formula to show that a vector field X on $\mathbb{R}^3$ has a flow (defined locally and for a short time) that preserves volume if and only if the divergence of X is everywhere zero. (Here, the divergence is the classical operator that takes a vector field with components f, g, h to the function $\partial_x f + \partial_y g + \partial_z h$, where x, y, z are the usual coordinates on $\mathbb{R}^3$.)

5. Let

$$\omega = \frac{-y dx + x dy}{(x^2 + y^2)^\alpha}$$

be a 1-form on $\mathbb{R}^2 - \{0\}$ with the usual coordinates (x, y), and for some $\alpha \in \mathbb{R}$. Consider $\int_\gamma \omega$, where $\gamma : S^1 \to \mathbb{R}^2 - \{0\}$ is a smooth map.

(a) For which $\alpha \in \mathbb{R}$ do we have $\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$ whenever γ_0 and γ_1 are *smoothly homotopic* (i.e., there exists a smooth map $F : S^1 \times [0, 1] \to \mathbb{R}^2 - \{0\}$ such that $\gamma_0(t) = F(t, 0), \gamma_1(t) = F(t, 1)$)?

(b) What are the possible values of $\int_\gamma \omega$ when α is chosen as in part (a)?

6. Let X and Y be connected CW-complexes, let $p \colon \tilde{X} \to X$ be a path connected covering space, and let $f \colon Y \to X$ be an arbitrary continuous map. Let

$$f^*(\tilde{X}) = \{(y, \tilde{x}) \mid f(y) = p(\tilde{x})\} \subset Y \times \tilde{X},$$

and consider the projection map $f^*(p) \colon f^*(\tilde{X}) \to Y, \ f^*(p)(y, \tilde{x}) = y$.

(a) Show that $f^*(p)$ is a covering map.

(b) Let $(y, \tilde{x}) \in f^*(\tilde{X})$, and let $x = f(y) = p(\tilde{x})$. If

$$f_* \pi_1(Y, y) \subset p_* \pi_1(\tilde{X}, \tilde{x}),$$

and the cover $p : \tilde{X} \to X$ is nontrivial, show that $f^*(\tilde{X})$ is disconnected.

7. Let $X = S^1 \times D^2$ with boundary $\partial X = S^1 \times S^1$. Compute the relative homology groups $H_k(X, \partial X; \mathbb{Z})$ for all k.

8. Let X be a CW complex and let $\tilde{X} \to X$ be a covering space. Let G be the group of deck transformations of $\tilde{X} \to X$.

(a) Show that for any k and for any abelian group M, the group G acts naturally on $H_k(\tilde{X}; M)$.

(b) Show that the map $p_* \colon H_k(\tilde{X}; M) \to H_k(X; M)$ factors through the quotient of $H_k(\tilde{X}; M)$ by the subgroup S generated by $m - g \cdot m$ for all $m \in H_k(\tilde{X}; M)$ and $g \in G$.

(c) Give an example for which the induced map $H_k(\tilde{X}; M)/S \to H_k(X; M)$ in (b) is not surjective.

9. (a) Find the homology groups $H_k(\mathbb{R}P^2; \mathbb{Z})$ for all k.

(b) Describe a cell decomposition for $\mathbb{R}P^2 \times \mathbb{R}P^2$. Use it to show (without appealing to the Künneth theorem) that $H_3(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z})$ is nontrivial.

10. Let G be a finite group and X a smooth manifold on which G acts smoothly. If the action of G on X is free (i.e., if $g \cdot x = x$ for some $x \in X$, then $g = 1$), then show that the natural quotient map

$$X \to X/G$$

is a covering map.