Geometry Topology Qualifying Exam, Fall 2018

(Q-1) Let M be a compact smooth n-manifold, and $f\colon M \to \mathbb{R}^n$ a smooth map. Let

$$S = \{p \in M \mid \operatorname{rank}(df_p) < n\}.$$

(a) Prove $S \neq \varnothing$.
(b) Prove $f(S) \subset \mathbb{R}^n$ has empty interior.

(Q-2) Let M_n be the space of $n \times n$ real matrices, viewed as the smooth manifold $\mathbb{R}^{n^2}$. Let M_n^k be the subset of matrices of rank k. Prove that M_n^k is a smooth submanifold of M_n. (Hint: First prove the subset of M_n^k where the top-left $k \times k$ minor is non-singular is a smooth submanifold M_n.)

(Q-3) Let θ be the restriction of

$$(x^2dx^1 - x^1dx^2) + (x^4dx^3 - x^3dx^4) + \cdots + (x^{2n}dx^{2n-1} - x^{2n-1}dx^{2n})$$

to the unit sphere $S^{2n-1} \subset \mathbb{R}^{2n}$. Prove $\ker(\theta)$ is a distribution on S^{2n-1}. Is it integrable?

(Q-4) Let M be a compact smooth 3-manifold and $\omega \in \Omega^1(M)$ a nowehere zero 1-form, so that $\ker(\omega)$ is an integrable distribution. Prove the following.

(a) $\omega \wedge d\omega = 0$.
(b) There exists some 1-form α with $d\omega = \alpha \wedge \omega$.
(c) $d\alpha \wedge \omega = 0$.

(Q-5) Let $M \subset \mathbb{R}^n$ be a compact $(n-1)$-dimensional submanifold, let $\iota\colon M \hookrightarrow \mathbb{R}^n$ be the inclusion map, and let $D \subset \mathbb{R}^n$ be the n-dimensional compact region with $\partial D = M$. Let $dV = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \in \Omega^n(\mathbb{R}^n)$ be the standard volume form on $\mathbb{R}^n$.

(a) Define $dA \in \Omega^{n-1}(M)$, the standard volume form on M, induced by the embedding ι.
(b) Prove $\iota^*(i_X dV) = \langle X, N\rangle\, dA$, for any smooth vector field X on $\mathbb{R}^n$. (Here, N is the unit normal vector field along M, pointing outward from D.)
(c) Prove

$$\int_D L_X(dV) = \int_M \langle X, N\rangle\, dA.$$

(d) Derive Gauss' Divergence Theorem from the case $n = 3$.

(Q-6) Can a finite rank free group have a finite index subgroup of smaller rank?

(Q-7) Prove that the covering map $S^n \to \mathbb{RP}^n$ induces an isomorphism on de Rham cohomology if and only if n is odd. What is the orientable double cover of $\mathbb{RP}^n$?

(Q-8) Assume the integral homology of a space is $\mathbb{Z}$ in grading 0, $\mathbb{Z}$ in grading 2, $\mathbb{Z}/2$ in grading 3, and 0 in all other gradings.

(a) What is its integral cohomology group?
(b) Construct a simply connected CW complex X with the given homology.
(c) Construct another simply connected CW complex Y with the same homology, which is not homotopy equivalent to X.

(Q-9) Let X be a connected CW-complex. Show that there is a natural isomorphism

$$\widetilde{H}_k(\Sigma X; M) \cong \widetilde{H}_{k-1}(X; M)$$

for all k and for all abelian groups M.

(Q-10) Let Y be a connected and simply connected CW-complex.

(a) Compute the fundamental group of $Y \vee S^1$.
(b) Describe the universal cover of $Y \vee S^1$, together with the action of the deck transformations.
(c) Describe all finite covers of $Y \vee S^1$, again with the action of the deck transformations.
(d) Describe what changes in the first two parts for $Y = \mathbb{RP}^2$.