QUALIFYING EXAM
Geometry/Topology
March 2018

Attempt all ten problems. Each problem is worth 10 points. Justify your answers carefully.

1. Suppose that M and N are connected smooth manifolds of the same dimension and $f : M \to N$ is a smooth submersion.

(a) Prove that if M is compact, then f is onto and f is a covering map.

(b) Give an example of a smooth submersion $f : M \to N$ such that M and N have the same dimension, N is compact, and f is onto, but f is not a covering map.

2. Let $\Phi_N, \Phi_S : \mathbb{R} \times S^2 \to S^2$ be two global flows on the sphere S^2. Show that there exist $\epsilon > 0$, a neighborhood U of the North pole, a neighborhood V of the South pole, and a global flow $\Phi : \mathbb{R} \times S^2 \to S^2$ such that $\Phi(t, q) = \Phi_N(t, q)$ for all $t \in (-\epsilon, \epsilon), q \in U$, and $\Phi(t, q) = \Phi_S(t, q)$ for all $t \in (-\epsilon, \epsilon), q \in V$.

3. For $n \geq 1$, consider the subset $X \subset \mathbb{CP}^{2n}$ given by

$$X = \{[z_0 : z_1 : \cdots : z_{2n}] \in \mathbb{CP}^{2n} \mid z_{n+1} = z_{n+2} = \cdots = z_{2n} = 0\}.$$

(a) Show that X is a smooth submanifold.

(b) Calculate the mod 2 intersection number of X with itself.

4. Suppose N is a smoothly embedded submanifold of a smooth manifold M. A vector field X on M is called tangent to N if $X_p \in T_pN \subset T_pM$ for all $p \in M$.

(a) Show that if X and Y are vector fields on M both tangent to N, then $[X, Y]$ is also tangent to N.

(b) Illustrate this principle by choosing two vector fields X, Y tangent to $S^2 \subset \mathbb{R}^3$ (such that $[X, Y]$ is not identically zero), computing $[X, Y]$ and checking that it is tangent to S^2.

5. A symplectic form on an eight-dimensional manifold is defined to be a closed two-form ω such that $\omega \wedge \omega \wedge \omega \wedge \omega$ is a volume form (that is, everywhere nonvanishing). Determine which of the following manifolds admit symplectic forms: (a) S^8; (b) $S^2 \times S^6$; (c) $S^2 \times S^2 \times S^2 \times S^2$.

6. Let U be a bounded open set in $\mathbb{R}^3$ with smooth boundary, and let V be a smooth vector field on $\mathbb{R}^3$. The classical divergence theorem expresses the triple integral $\iiint_V \operatorname{div} V d(\text{vol})$ as a surface integral over the boundary of V. State this theorem, and show how it can be obtained as a particular case of Stokes' Theorem for differential forms.

7. Let M and N be smooth, connected, orientable n-manifolds for $n \geq 3$, and let $M \# N$ denote their connect sum.

(a) Compute the fundamental group of $M \# N$ in terms of that of M and of N (you may assume that the basepoint is on the boundary sphere along which we glue M and N).

(b) Compute the homology groups of $M\#N$. (You may use without proof that $H_n(-;\mathbb{Z})$ of a connected orientable n-manifold is always isomorphic to $\mathbb{Z}$).

(c) For part (a), what changes if $n = 2$? Use this to describe the fundamental groups of orientable surfaces.

8. Determine all of the possible degrees of maps $S^2 \to S^1 \times S^1$.

9. Point S^2 via the south pole, and consider the Cartesian product $S^2 \times S^2$.

(a) Describe a cell structure on $S^2 \times S^2$ that is compatible with the inclusion of

$$S^2 \vee S^2 \hookrightarrow S^2 \times S^2$$

as those pairs where one coordinate is the south pole.

(b) Let X be $(S^2 \times S^2) \cup_{S^2} D^3$, where we attach the 3-disk via the map

$$S^2 \to S^2 \vee S^2$$

which crushes a great circle connecting the north and south poles. Compute the homology groups of X.

10. Let X be a semi-locally simply connected space and let $\tilde{X} \to X$ be the universal cover.

(a) Show that any map $\sigma\colon \Delta^n \to X$ lifts to a map $\tilde{\sigma}\colon \Delta^n \to \tilde{X}$, where Δ^n is the standard n-simplex.

(b) Show that if $\tilde{\sigma}_1, \tilde{\sigma}_2\colon \Delta^n \to \tilde{X}$ are two lifts of σ, then there is an element g of the fundamental group of X such that $g \circ \tilde{\sigma}_1 = \tilde{\sigma}_2$, where we view g as an automorphism of $\tilde{X}$ via the deck transformations.