QUALIFYING EXAM
Geometry/Topology
September 2019

Attempt all ten problems. Each problem is worth 10 points. Justify your answers carefully.

1. State the classical Divergence Theorem (also called Gauss' Divergence Theorem) for a compact 3-dimensional submanifold of $\mathbb{R}^3$ with smooth boundary. Derive it from Stokes' Theorem for differential forms.

2. Compute $H_*(\mathbb{RP}^{n+m}/\mathbb{RP}^n;\mathbb{Z})$ as a function of n and m. Here we are viewing $\mathbb{RP}^n \subset \mathbb{RP}^{n+m}$ induced from the inclusion
$$\mathbb{R}^{n+1} \subset \mathbb{R}^{n+m+1}, \quad (x_1,\ldots,x_{n+1}) \mapsto (x_1,\ldots,x_{n+1},0,\ldots,0).$$

3. For which $n > 0$ does the real projective space $\mathbb{RP}^n$ admit a nowhere-vanishing vector field? If a nowhere-vanishing vector field exists, give an explicit one.

4. Let $X = S^1 \times S^1$ and let Y be the quotient of $X \times [0,1]$ by the relation
$$\big((x,y),0\big) \sim \big((y,x),1\big).$$
Compute $H_*(Y;\mathbb{Z})$.

5. A vector field X on a Lie group G is *left-invariant* if $(L_g)_*X = X$ for all $g \in G$, where $L_g : G \to G$, $g' \mapsto gg'$, is left multiplication by g. Show that if X, Y are left-invariant vector fields, then so is their Lie bracket $[X,Y]$ as vector fields. You must prove any fact about Lie brackets that you use.

6. Let $Z(P)$ be the zero set of a degree d homogeneous polynomial $P(z_0,z_1,z_2)$ in $\mathbb{CP}^2$ with respect to the homogeneous coordinates $[z_0 : z_1 : z_2]$. Assuming P has no repeated factors, give necessary and sufficient conditions on P for $Z(P)$ to be smooth.

7. If $P(z_0,z_1,z_2) = z_0^d + z_1^d + z_2^d$ from the previous problem, then compute the Euler characteristic of $Z(P)$. You must show all your work.

8. Let $X = \{N, S, E, W\}$ with the topology given by
$$\Big\{\emptyset, \{E\}, \{W\}, \{E,W\}, \{N,E,W\}, \{S,E,W\}, X\Big\}.$$
For each n, find a path-connected degree n cover. Describe the universal cover.

9. (a) If
$$0 \to A \to B \to C \to 0$$
is a short exact sequence of chain complexes, show how to get the boundary map in the associated long exact sequence.

(b) Compute the boundary map when the short exact sequence is the result of tensoring the chain complex

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{5} \mathbb{Z} \to 0 \to \dots$$

with the short exact sequence

$$0 \to \mathbb{Z}/5 \xrightarrow{5} \mathbb{Z}/25 \to \mathbb{Z}/5 \to 0.$$

10. Let X be a path-connected, locally path-connected, semi-locally simply-connected space and let $\tilde{X} \to X$ be the universal cover.

(a) Given $x_0 \in X$, explain how $\pi_1(X, x_0)$ can be viewed as the set of deck transformations of $\tilde{X}$.

(b) Show that any map $\sigma \colon \Delta^n \to X$ lifts to a map $\tilde{\sigma} \colon \Delta^n \to \tilde{X}$, where Δ^n is the standard n-simplex.

(c) Show that the action of $\pi_1(X, x_0)$ on the set of maps $\tilde{\sigma} \colon \Delta^n \to \tilde{X}$ is free (i.e., if $g \in \pi_1(X, x_0)$ and $g \circ \tilde{\sigma} = \tilde{\sigma}$, then $g = \mathrm{id}$).

(d) Show that if $\tilde{\sigma}_1, \tilde{\sigma}_2 \colon \Delta^n \to \tilde{X}$ are two lifts of σ, then there exists $g \in \pi_1(X, x_0)$ such that $g \circ \tilde{\sigma}_1 = \tilde{\sigma}_2$.