QUALIFYING EXAM
Geometry/Topology
March 2019

Attempt all ten problems. Each problem is worth 10 points. Justify your answers carefully.

1. Let M be a smooth manifold. Show that there exists a smooth proper map $\rho : M \to \mathbb{R}$.

2. A smooth manifold Y of dimension n is called *parallelizable* if there exist n vector fields $v^1, \ldots, v^n$ on Y such that at every $y \in Y$, the tangent vectors $v^1_y, \ldots, v^n_y$ are linearly independent.

Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be a smooth function. Suppose that $0 \in \mathbb{R}$ is a regular value, and let M be the smooth manifold $f^{-1}(\{0\})$. Show that $M \times S^1$ is parallelizable.

3. Show that the antipodal map $A : S^n \to S^n$, $A(x) = -x$ is homotopic to the identity if and only if n is odd. (Feel free to use Lefschetz theory if you like.)

4. For a vector field X on a smooth manifold M, we denote $L_X : \Omega^k(M) \to \Omega^k(M)$ denote the Lie derivative of X acting on k-forms.

Prove that, for any vector fields X and Y, we have the Lie bracket identity

$$[L_X, L_Y] = L_{[X,Y]}.$$

5. Show that a closed 1-form ω on a manifold M is exact if and only if $\int_{S^1} f^*\omega = 0$ for every smooth map $f : S^1 \to M$.

6. Let $f : X \to Y$ be a smooth, finite covering map between smooth manifolds. Show that the induced map on deRham cohomology

$$f^* : H^n_{\mathrm{dR}}(Y; \mathbb{R}) \to H^n_{\mathrm{dR}}(X; \mathbb{R})$$

is injective.

7. Let $X = [0,1]$ and $A = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}, n \geq 1\}$. Show that $H_1(X, A)$ is not isomorphic to $H_1(X/A)$.

8. (a) Show that any continuous map $\mathbb{R}P^2 \to S^1 \times S^1$ is nullhomotopic.

(b) Find, with proof, a continuous map $S^1 \times S^1 \to \mathbb{R}P^2$ that is not nullhomotopic.

9. Let W be the space obtained by attaching two 2-cells to S^1, one by the map $z \to z^4$ and the other by the map $z \to z^7$. (Here, $z = e^{i\theta}$ is the coordinate on $S^1 = \partial D^2$.)

(a) Compute the homology groups of W with $\mathbb{Z}$ coefficients.

(b) Is W homotopy equivalent to S^2?

10. Suppose M^n is a compact, connected, orientable topological n-manifold with boundary a rational homology sphere, i.e. with $H_*(\partial M; \mathbb{Q}) \cong H_*(S^{n-1}; \mathbb{Q})$.

(a) Assuming n is odd, use Poincaré duality (with $\mathbb{Q}$ coefficients) to show that M has Euler characteristic $\chi(M) = 1$.

(b) Assuming $n \equiv 2 \pmod 4$, show that the Euler characteristic $\chi(M)$ is odd.