Attempt all ten problems. Each problem is worth 10 points. You must fully justify your answers.

1. Let $x_1, x_2, \ldots, x_k$ and $y_1, y_2, \ldots, y_k$ be two sets of distinct points in a connected smooth manifold M with $dim(M) > 1$, and $v_1, v_2, \ldots, v_k$ and $w_1, w_2, \ldots, w_k$ be the corresponding two sets of non-zero tangent vectors at these points. Show that there is a diffeomorphism f of M such that $f(x_i) = y_i$ and $df_{x_i}(v_i) = w_i$ for $i = 1, \ldots, k$.

2. Let M be a smooth manifold of dimension n. Let $T^*M := \sqcup_{m \in M} T_m^* M$ be the cotangent bundle, where $T_m^* M$ is the dual of the tangent space $T_m M$, and let $\pi : T^*M \to M$ be the natural projection such that $\pi(\phi) = m$ for $\phi \in T_m^* M$. Let $x = (x_1, \ldots, x_n)$ be local co-ordinates on $U \subset M$. Then we endow $\pi^{-1}(U)$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, such that the element $\phi \in \pi^{-1}(U)$ with $\pi(\phi) = m$ is written as $\sum_i y_i dx_i(m)$.

   (a) Show that T^*M is a smooth manifold with respect to the local coordinate charts defined above.

   (b) Define the 1-form λ on the cotangent bundle T^*M as follows: for any tangent vector $v \in T_\phi(T^*M)$ at $\phi \in T^*M$, we set $\lambda(\phi)(v) = \phi(d\pi(v))$.
   Find the explicit expression of λ with respect to the above local coordinate chart. Use this to show that λ is smooth.

   (c) Find the explicit expression of $d\lambda$ and its k-th exterior powers for all $k \geq 2$ with respect to the local coordinate chart above. Use this to show that T^*M is orientable.

3. Let M be a smooth manifold with smooth boundary ∂M and N be a smooth manifold without boundary. Assume that $f : M \to N$ is smooth (this includes smoothness at points of ∂M) so that the tangent map $df_x : T_x M \to T_{f(x)} N$ is well-defined (including at points of ∂M). Let $y \in N$ be a regular value for both f and $f|_{\partial M}$.

   (a) Show that $M_1 := f^{-1}(y)$, if not empty, is a smooth submanifold with boundary in M such that the boundary $\partial M_1 = (f|_{\partial M})^{-1}(y) = M_1 \cap \partial M$ is a submanifold of ∂M.

   (b) If we only assume that y is a regular value for f but not for $f|_{\partial M}$, does the conclusion of (a) still hold?

4. Let S be a closed subset of a smooth manifold M that has a second countable topological basis. Show that for any positive integer n, there is a smooth map $f : M \to \mathbb{R}^n$ such that $S = f^{-1}(0)$.

5. Let $M, N \subset \mathbb{R}^{p+1}$ be two compact, smooth, oriented submanifolds (without boundary) of dimensions m and n, respectively, such that $m + n = p$, and suppose that $M \cap N = \varnothing$. Let $l(M, N)$ be the degree of the map

$$\lambda : M \times N \to S^p, \quad \lambda(x, y) = \frac{x - y}{\|x - y\|}.$$

(a) Show that $l(M, N) = (-1)^{(m+1)(n+1)} l(N, M)$.

(b) Show that if M is the boundary of an oriented submanifold $W \subset \mathbb{R}^{p+1}$ which is disjoint from N, then $l(M, N) = 0$.

6. Let X be a topological space and let $Y = X \times [-1, 1]/\sim$, where

$$(x, -1) \sim (x', -1) \quad \text{for all } x, x' \in X,$$
$$(x, 1) \sim (x', 1) \quad \text{for all } x, x' \in X.$$

Describe the relationship between the homology groups of X and Y.

7. (a) Describe a cell decomposition for $X = \mathbb{RP}^4$ such that its 2-skeleton $X^{(2)} = \mathbb{RP}^2$. (This means that X is obtained from $X^{(2)}$ by attaching only 3- and 4-dimensional cells.) Include a careful description of the attaching maps.

(b) Use your cell decomposition to compute $H_k(X; \mathbb{Z})$ and $H_k(X, X^{(2)}; \mathbb{Z})$ for all $k \geq 0$.

8. List all the 3-sheeted connected covering spaces of $S^1 \vee S^1$. Which ones in the list are not normal?

9. Let Σ_5 be a compact oriented surface of genus 5 without boundary. Does there exist an immersion $f : T^2 \to \Sigma_5$? Justify your answer.

10. Show that the Euler characteristic of the special linear group $SL(n, \mathbb{R})$ with $n > 1$ is zero. Here for a topological space X its Euler characteristic is $\chi(X) := \Sigma_i (-1)^i \mathrm{rank}\, (H_i(X))$, assuming that $\Sigma_i \mathrm{rank}\, (H_i(X)) < \infty$.