Qualifying Exam
Geometry/Topology

Attempt all ten problems. Each problem is worth 10 points. Justify your answers carefully.

(Q-1) Suppose $f\colon M \to N$ is a smooth map between smooth manifolds, and is smoothly homotopic to a locally constant map. Prove $f^*\omega$ is exact for any closed differential k-form ω on N (with $k > 0$).

(Q-2) Let M be a smooth 4-dimensional manifold. A symplectic form is a closed 2-form ω on M such that $\omega \wedge \omega$ is a nowhere vanishing 4-form.
  (a) Construct a symplectic form on $\mathbb{R}^4$.
  (b) Show that there are no symplectic forms on the unit sphere S^4.

(Q-3) Consider the differential form $\omega = xdy - ydx - dz$ in $\mathbb{R}^3$ with coordinates (x, y, z). Prove that $f\omega$ is not closed for any nowhere zero smooth function $f\colon \mathbb{R}^3 \to \mathbb{R}$.

(Q-4) For any two smooth vector fields X, Y on a smooth manifold M, prove the formula

$$[L_X, i_Y] = i_{[X,Y]}$$

where L_X denotes the Lie derivative and i_X is the contraction of vector field acting on differential forms.

(Q-5) Show that the complex projective space $\mathbb{C}\mathrm{P}^{2n}$ does not cover any manifold except itself.

(Q-6) Show that any continuous map from $S^2 \times S^2$ to $\mathbb{C}\mathrm{P}^2$ must be of even degree.

(Q-7) Prove that the relative homology groups $H_k(X, x)$ for different choices of basepoint x can be naturally identified with each other. That is, for every $k \geq 0$, every space X, and all pairs of points $x, y \in X$ (not necessarily in the same connected component), construct isomorphisms $\eta^X_{x,y}\colon H_k(X, x) \to H_k(X, y)$ satisfying
  (a) $\eta^X_{x,x} = \mathrm{Id}$ for all $x \in X$.
  (b) $\eta^X_{y,z} \circ \eta^X_{x,y} = \eta^X_{x,z}$ for all $x, y, z \in X$.
  (c) $f_* \circ \eta^X_{x,y} = \eta^Y_{f(x),f(y)} \circ f_*$ for all $x, y \in X$ and all continuous maps $f\colon X \to Y$.
  (Hint: You might consider proving the case $k \geq 1$ first.)

(Q-8) Assume the integral homology of a finite CW complex X is $\mathbb{Z}$ in grading 0, $\mathbb{Z}/2$ in grading 2, $\mathbb{Z}/3$ in grading 3, and 0 in all other gradings. What is the cohomology of X with $\mathbb{Z}/6$ coefficients? Can you give an example of such a space X?

(Q-9) Consider the following group with $2n$ generators and 1 relation

$$G_n = \langle a_1, b_1, a_2, b_2, \ldots, a_n, b_n \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1} \rangle$$

For which pairs (m, n) does G_n contain a finite index subgroup isomorphic to G_m?

(Q-10) Let D^2 be the unit disk in $\mathbb{C}$, and let $S^1 = \partial D^2$. Let $X = D^2 \times S^1 \times \{0, 1\}/\sim$ where

$$(x, y, 0) \sim (xy^5, y, 1)$$

for all $x, y \in S^1$. Compute the homology groups of X.