

All nine problems have equal value. **Good luck!**

1. Let $(\varphi_e)_{e \in \mathbb{N}}$ be some canonical listing of the partial recursive functions. Call a total function f “large” if $f(e) > \varphi_e(0)$ for all e for which $\varphi_e(0)$ is defined. Show that if f is large then it has degree at least $\mathbf{0}'$.

2. Let Σ be a set of first-order sentences and let \mathfrak{M} be a structure. Show that if every finitely generated substructure of \mathfrak{M} has an extension to a model of Σ then so does \mathfrak{M} .

3. Let \mathfrak{A} be a model for a finite language \mathcal{L} containing the predicate symbols P , Q , and R . Let

$$\begin{aligned} M &= \{a \mid \mathfrak{A} \models P(a)\} \\ K &= \{(a, b, c) \mid \mathfrak{A} \models Q(a, b, c)\} \\ L &= \{(a, b, c) \mid \mathfrak{A} \models R(a, b, c)\}. \end{aligned}$$

Suppose that the set M together with the relations K and L is isomorphic to \mathbb{N} together with addition and multiplication (as ternary relations). Prove that the theory of \mathfrak{A} is undecidable.

4. Recall that a formula $\varphi(x_1, \dots, x_n)$ is *complete* (in a complete theory T) if for every formula $\psi(x_1, \dots, x_n)$ either

$$T \models \varphi \rightarrow \psi, \text{ or } T \models \varphi \rightarrow \neg\psi.$$

A model \mathfrak{M} is said to be *atomic* if every $\vec{a} \in \mathfrak{M}^{<\omega}$ satisfies a complete formula in the theory of \mathfrak{M} .

Give an example of a countable complete theory that has an atomic model of size \aleph_1 but is not \aleph_0 -categorical. Justify your answer (prove that your example works).

5. Recall that a model \mathfrak{M} is ω -saturated if any consistent type is realized. More precisely: for $\vec{a} \in \mathfrak{M}^{<\omega}$, $\Sigma(\vec{a}, x)$ is a *consistent* type in the variable x if for any finite

$$\{\varphi_1(\vec{a}, x), \dots, \varphi_n(\vec{a}, x)\} \subseteq \Sigma(\vec{a}, x)$$

there is some $b \in \mathfrak{M}$ with $\mathfrak{M} \models \varphi_i(\vec{a}, b)$ for all $i \in \{1, 2, \dots, n\}$. \mathfrak{M} is ω -saturated if for any $\vec{a} \in \mathfrak{M}^{<\omega}$ and any consistent type $\Sigma(\vec{a}, x)$ there is some actual $c \in \mathfrak{M}$ with $\mathfrak{M} \models \varphi(\vec{a}, c)$ for all $\varphi(\vec{a}, x) \in \Sigma(\vec{a}, x)$.

Show that if a countable complete theory T has only \aleph_1 -many non-isomorphic countable models then it has an ω -saturated model of size less than \aleph_2 .

6. Let ZFC^- be all the axioms of ZFC except the power set axiom. Recall that for each ordinal α there is an ordinal $\beta > \alpha$ with

$$L_\beta \models \text{ZFC}^-.$$

For each α let $\pi(\alpha)$ be the least ordinal above α with

$$L_{\pi(\alpha)} \models \text{ZFC}^-.$$

Show that there is some $\alpha < \omega_1$ with

$$L_{\pi(\alpha)} \models “\alpha > \omega_2”.$$

7. (a) Show that if κ is a regular uncountable cardinal then for all $A \in L_\kappa$ there is $\alpha < \kappa$ with

$$A \in L_\alpha$$

and

$$L_\alpha \prec L_\kappa$$

($L_\alpha \prec L_\kappa$ stands for " L_α is an elementary substructure of L_κ ").

(b) A cardinal κ is *weakly inaccessible* if it is regular, uncountable, and for each $\gamma < \kappa$ we have $\gamma^+ < \kappa$. Show that if κ is weakly inaccessible then

$$L_\kappa \models \text{ZFC}.$$

(c) Assuming $\text{ZFC} + \text{Con}(\text{ZFC})$ to be consistent, show that $\text{ZFC} + \text{Con}(\text{ZFC})$ does not prove $\text{Con}(\text{ZFC} + \text{there exists a weakly inaccessible cardinal})$. (For a recursively axiomatized theory T , recall that $\text{Con}(T)$ is the canonical formula expressing the consistency of T .)

8. Assume ZFC is consistent. Show that there is a model \mathfrak{N} of ZFC containing no (genuine) well-orderings of an infinite set. That is to say, whenever $a, w \in \mathfrak{N}$ with

$$A = \{x \in \mathfrak{N} \mid \mathfrak{N} \models "x \in a"\}$$

infinite, the relation

$$R = \{\langle x, y \rangle \mid \mathfrak{N} \models "\langle x, y \rangle \in w"\}$$

is not a well-ordering of A .

9. Let $(W_e)_{e \in \mathbb{N}}$ be a canonical enumeration of the r.e. subsets of \mathbb{N} . Let A be the set of e for which W_e contains an arithmetical progression of the kind $\{2^h + n2^{h+1} \mid n \in \mathbb{N}\}$.

(a) Show that A is Σ_3 .

(b) Show that every Σ_3 subset of \mathbb{N} is many-one reducible to A .