Qualifying Examination
LOGIC
Fall 2002

All problems have equal value. Solution of four problems will ensure an M.A. pass.

1. As usual, for each number $e \in \mathbb{N}$,

$$W_e = \{x \mid \phi_e(x) \downarrow\},$$

where $\phi_0, \phi_1, \ldots$ is a standard enumeration of all recursive partial functions.

(1a) Prove that the set

$$A = \{e \mid W_e \text{ is finite}\}$$

is in $\Sigma_2 \setminus \Pi_2$.

(1b) Classify in the arithmetical hierarchy the set

$$B = \{e \mid W_e \text{ is finite and has an even number of elements}\};$$

i.e., find some n such that $B \in \Sigma_n \setminus \Pi_n$ or $B \in \Pi_n \setminus \Sigma_n$.

2. **(2a)** Prove that for every recursive partial function $f(x, y)$, there is some recursive partial function $g(x)$ such that

(A) $$g(x) \downarrow \iff (\exists y)[f(x,y) \downarrow],$$

(B) $$(\exists y)[f(x,y) \downarrow] \implies f(x, g(x)) \downarrow .$$

(2b) Show that we cannot strengthen (2a) by replacing (B) by the stronger

(B') $$(\exists y)[f(x,y) \downarrow] \implies g(x) = \text{the least } y \text{ such that } f(x,y) \downarrow .$$

3. For each sentence θ in the language of Peano arithmetic PA, let

$$\ulcorner\theta\urcorner = \text{the (formal) numeral of the Gödel number of } \theta,$$

and let $\mathrm{Pr}(n)$ be a formula with one free variable which expresses the relation of *provability* in Peano arithmetic, so that (in particular), for each sentence θ,

$$(\mathbb{N}, 0, 1, +, \cdot) \models \mathrm{Pr}(\ulcorner\theta\urcorner) \iff \mathrm{PA} \vdash \theta.$$

Consider the following four sentences which can be constructed from an arbitrary sentence θ:

(a) $\theta \to \mathrm{Pr}(\ulcorner\theta\urcorner)$
(b) $\mathrm{Pr}(\ulcorner\theta\urcorner) \to \theta$
(c) $\mathrm{Pr}(\ulcorner\theta\urcorner) \to \mathrm{Pr}(\ulcorner\mathrm{Pr}(\ulcorner\theta\urcorner)\urcorner)$
(d) $\mathrm{Pr}(\ulcorner\mathrm{Pr}(\ulcorner\theta\urcorner)\urcorner) \to \mathrm{Pr}(\ulcorner\theta\urcorner)$

Determine which of these four sentences are provable in PA (for every choice of θ), and *justify three of your answers* by appealing, if necessary, to standard theorems which are proved in 220. (One of the answers is more difficult to justify than the others.)

4. Assume $V = L$, let

$$\lambda = \aleph_\omega,$$

and prove that L_λ has the Σ_1-reflection property.

In detail, this means that if

$$\theta(x, y) \equiv (\exists x_1)(\exists x_2) \cdots (\exists x_n)\phi(x, y)$$

where $\phi(x, y)$ is a *bounded* formula in which all quantifiers occur in one of the forms

$$(\exists y \in z) \text{ or } (\forall y \in z),$$

and if for some $a \in L_\lambda$,

$$L_\lambda \models (\forall x \in a)(\exists y)\theta(x, y),$$

then there is some $b \in L_\lambda$ such that

$$L_\lambda \models (\forall x \in a)(\exists y \in b)\theta(x, y).$$

5. Suppose $V = L$. *True or false*: if α is an ordinal such that

$$L_\alpha \models \text{ZFC},$$

then α is a strongly inaccessible cardinal. (You must prove your answer.)

6. A *tree* on a set A is a set $T \subseteq A^{<\omega}$ of finite sequences from A which is closed under initial segments; an *infinite branch* of a tree T is any function $\alpha : \mathbb{N} \to A$ such that for all n, $\langle \alpha(0) < \ldots, \alpha(n-1) \rangle \in T$; and T is *finitely splitting* if for each $u \in T$ there are only finitely many (perhaps 0) one-point extensions of u in T. Prove the following

König's Lemma. Every infinite, finitely splitting tree T has an infinite branch.

7. Recall that a model $\mathfrak{M}$ of a complete (first-order) theory T is *atomic*, if for every finite sequence $\vec{a} \in M^n$ of length n, there is a formula $\phi(\vec{v})$ with n free variables such that

$$\mathfrak{M} \models \phi(\vec{a}),$$

and for every $\psi(\vec{v})$,

$$\text{either } \mathfrak{M} \models (\forall \vec{v})[\phi(\vec{v}) \to \psi(\vec{v})] \text{ or } \mathfrak{M} \models (\forall \vec{v})[\phi(\vec{v}) \to \neg\psi(\vec{v})]$$

(7a). Does there exist a countable, complete theory with an atomic model of size $\aleph_0$ but no atomic model of size $\aleph_1$?

(7b). Does there exist a countable, complete theory with an atomic model of size $\aleph_1$ but no atomic model of size $\aleph_0$?

8. Let $\mathfrak{M} = (\mathbb{Z}, S)$ be the model with underlying set the integers and the successor function $S(x) = x + 1$ as the only non-logical constant. Is $\text{Th}(\mathfrak{M})$ finitely axiomatizable?

9. Let $T = \text{Th}(\mathbb{R}, <, \mathbb{Z})$ be the theory of the real numbers, with the usual ordering and a distinguished predicate for the integers. Is T $\aleph_0$-categorical? (You must prove your answer.)