W 02

Qualifying Exam LOGIC ~~Fall 2001~~

All nine problems have equal value. **Good luck!**

1. Let $(\varphi_e)_{e\in\mathbb{N}}$ be some canonical listing of the partial recursive functions. Call a total function f "large" if $f(e) > \varphi_e(0)$ for all e for which $\varphi_e(0)$ is defined. Show that if f is large then it has degree at least $\mathbf{0}'$.

2. Let Σ be a set of first-order sentences and let $\mathfrak{M}$ be a structure. Show that if every finitely generated substructure of $\mathfrak{M}$ has an extension to a model of Σ then so does $\mathfrak{M}$.

3. Let $\mathfrak{A}$ be a model for a finite language $\mathcal{L}$ containing the predicate symbols P, Q, and R. Let

$$\begin{aligned} M &= \{a \mid \mathfrak{A} \models P(a)\} \\ K &= \{(a,b,c) \mid \mathfrak{A} \models Q(a,b,c)\} \\ L &= \{(a,b,c) \mid \mathfrak{A} \models R(a,b,c)\}. \end{aligned}$$

Suppose that the set M together with the relations K and L is isomorphic to $\mathbb{N}$ together with addition and multiplication (as ternary relations). Prove that the theory of $\mathfrak{A}$ is undecidable.

4. Recall that a formula $\varphi(x_1,\ldots,x_n)$ is *complete* (in a complete theory T) if for every formula $\psi(x_1,\ldots,x_n)$ either

$$T \models \varphi \rightarrow \psi, \text{ or } T \models \varphi \rightarrow \neg\psi.$$

A model $\mathfrak{M}$ is said to be *atomic* if every $\vec{a} \in \mathfrak{M}^{<\omega}$ satisfies a complete formula in the theory of $\mathfrak{M}$.

Give an example of a countable complete theory that has an atomic model of size $\aleph_1$ but is not $\aleph_0$-categorical. Justify your answer (prove that your example works).

5. Recall that a model $\mathfrak{M}$ is *ω-saturated* if any consistent type is realized. More precisely: for $\vec{a} \in \mathfrak{M}^{<\omega}$, $\Sigma(\vec{a},x)$ is a *consistent* type in the variable x if for any finite

$$\{\varphi_1(\vec{a},x),\ldots,\varphi_n(\vec{a},x)\} \subseteq \Sigma(\vec{a},x)$$

there is some $b \in \mathfrak{M}$ with $\mathfrak{M} \models \varphi_i(\vec{a},b)$ for all $i \in \{1,2,\ldots,n\}$. $\mathfrak{M}$ is *ω-saturated* if for any $\vec{a} \in \mathfrak{M}^{<\omega}$ and any consistent type $\Sigma(\vec{a},x)$ there is some actual $c \in \mathfrak{M}$ with $\mathfrak{M} \models \varphi(\vec{a},c)$ for all $\varphi(\vec{a},x) \in \Sigma(\vec{a},x)$.

Show that if a countable complete theory T has only $\aleph_1$-many non-isomorphic countable models then it has an ω-saturated model of size less than $\aleph_2$.

6. Let ZFC$^-$ be all the axioms of ZFC except the power set axiom. Recall that for each ordinal α there is an ordinal $\beta > \alpha$ with

$$L_\beta \models \text{ZFC}^-.$$

For each α let $\pi(\alpha)$ be the least ordinal above α with

$$L_{\pi(\alpha)} \models \text{ZFC}^-.$$

Show that there is some $\alpha < \omega_1$ with

$$L_{\pi(\alpha)} \models \text{“}\alpha > \omega_2\text{”}.$$

7. (a) Show that if κ is a regular uncountable cardinal then for all $A \in L_\kappa$ there is $\alpha < \kappa$ with

$$A \in L_\alpha$$

and

$$L_\alpha \prec L_\kappa$$

($L_\alpha \prec L_\kappa$ stands for "L_α is an elementary substructure of L_κ").

(b) A cardinal κ is ***weakly inaccessible*** if it is regular, uncountable, and for each $\gamma < \kappa$ we have $\gamma^+ < \kappa$. Show that if κ is weakly inaccessible then

$$L_\kappa \models \text{ZFC}.$$

(c) Assuming ZFC + Con(ZFC) to be consistent, show that ZFC + Con(ZFC) does not prove Con(ZFC + there exists a weakly inaccessible cardinal). (For a recursively axiomatized theory T, recall that Con(T) is the canonical formula expressing the consistency of T.)

8. Assume ZFC is consistent. Show that there is a model $\mathfrak{N}$ of ZFC containing no (genuine) well-orderings of an infinite set. That is to say, whenever $a, w \in \mathfrak{N}$ with

$$A = \{x \in \mathfrak{N} \mid \mathfrak{N} \models \text{"}x \in a\text{"}\}$$

infinite, the relation

$$R = \{\langle x, y\rangle \mid \mathfrak{N} \models \text{"}\langle x, y\rangle \in w\text{"}\}$$

is not a well-ordering of A.

9. Let $(W_e)_{e\in\mathbb{N}}$ be a canonical enumeration of the r.e. subsets of $\mathbb{N}$. Let A be the set of e for which W_e contains an arithmetical progression of the kind $\{2^h + n2^{h+1} \mid n \in \mathbb{N}\}$.

(a) Show that A is Σ_3.

(b) Show that every Σ_3 subset of $\mathbb{N}$ is many-one reducible to A.