

All questions have equal value.

1. For this problem, work in ZF (ZFC minus the Axiom of Choice). If  $\kappa$  is a cardinal number and  $X$  is a set, then  $\mathcal{P}_\kappa(X)$  is the set of all subsets of  $X$  of size  $< \kappa$ . Suppose that  $f : \mathcal{P}_{\omega_1}(\mathbb{R}) \rightarrow \mathbb{R}$  is one-to-one. Prove that there exists a sequence of  $\omega_1$  distinct reals.
2. A subset  $X$  of a limit ordinal  $\alpha$  is *stationary* in  $\alpha$  if  $X$  meets every closed, unbounded subset of  $\alpha$ . Let  $\kappa$  be a regular cardinal and let  $X \subseteq \kappa$  be stationary in  $\kappa$ . Let  $M$  be a transitive class model of ZFC such that  $X \in M$ . Prove that  $X$  is stationary in  $\kappa$  in  $M$ .
3. Assume  $V = L$ . Define  $\langle A_\alpha \mid \alpha < \omega_1 \rangle$  as follows. Let  $A_\alpha$  be the  $<_L$ -least  $A \subseteq \alpha$  such that  $(\forall \beta < \alpha) A \cap \beta \neq A_\beta$  if such an  $A$  exists and let  $A_\alpha = \emptyset$  otherwise. Prove that for all  $A \subseteq \omega_1$  there exists an  $\alpha < \omega_1$  such that  $A \cap \alpha = A_\alpha$ .
4. As with problem 1, work in ZF. Let  $\text{AC}^{\text{fin}}$  be the restriction of the Axiom of Choice to collections of *finite* sets. Prove that the Compactness Theorem of model theory implies  $\text{AC}^{\text{fin}}$ .
5. Let  $S(n) = n + 1$  for  $n \in \omega$ . Prove that the theory of  $(\omega, S)$  is not finitely axiomatizable.
6. Let  $\kappa = \omega_1$  and let  $T = \text{Th}(V_\kappa, \in)$ . Prove that there is no saturated countable model of  $T$ .
7. Let  $A$  be an infinite recursively enumerable set. Show that  $\{e \mid W_e = A\}$  is many-one complete for  $\Pi_2$ . (Here  $W_e$  is the  $e$ th r.e. set in some standard enumeration.)
8. Let  $\text{Prov}(v_1, v_2)$  represent in Peano Arithmetic (PA) the set of all pairs  $(a, b)$  such that  $a$  is the Gödel number of a sentence  $\tau$  and  $b$  is the Gödel number of a proof of  $\tau$  from the axioms of PA. Let  $\sigma$  be gotten from the Fixed Point Lemma applied to  $\forall v_2 \neg \text{Prov}(v_1, v_2)$ . In other words, let  $\sigma$  be a sentence such that  $\text{PA} \vdash (\sigma \leftrightarrow \forall v_2 \neg \text{Prov}(k, v_2))$ , where  $k$  is the Gödel number of  $\sigma$ . Let  $T$  be the theory gotten from PA by adding  $\neg \sigma$  as an axiom. Show that  $T$  is  $\omega$ -inconsistent: that there is a formula  $\psi(v_1)$  such that  $T \vdash \exists v_1 \psi(v_1)$  and  $T \vdash \neg \psi(\mathbf{n})$  for each numeral  $n$ .