Qualifying Exam **LOGIC** **Winter 2003**

All eight questions have equal value. Good luck!

1. A linear order $(S, \prec)$ is **illfounded** if there is an infinite sequence $\langle s_i \mid i \in \omega \rangle$ of elements of S so that $s_{i+1} \prec s_i$ for each $i \in \omega$.

Let φ be a sentence in the language with one binary relation symbol $\prec$. Suppose φ is true in an infinite illfounded linear order. Prove that φ is true in an uncountable illfounded linear order.

2. For each complete theory T in the language of set theory let $A_T = \{\alpha < \omega_1 \mid \mathrm{L}_\alpha \models T\}$. Prove that there is a complete theory T so that A_T is uncountable.

3. Let U be the set of all functions f so that: $\mathrm{dom}(f) = \omega$; and (for every $n \in \omega$) $f(n)$ belongs to ω_n. Assuming the GCH, prove that $\mathrm{card}(U) = \aleph_{\omega+1}$.

4. (a) For each formula φ in the language of set theory, show that ZFC proves $\varphi \rightarrow CON(\varphi)$.

(b) Show that ZFC is not finitely axiomatizable.

5. Let $\langle \phi_e \mid e < \omega \rangle$ be a standard enumeration of all the recursive partial functions. Fix a total recursive function f. Let $B = \{e \mid \phi_e = f\}$. Prove that B is Π_2 complete.

6. Let $\varphi(x_1, \ldots, x_n)$ be a Σ_1 formula in the language of set theory. Suppose $\mathrm{V} = \mathrm{L}$. Let $a_1, \ldots, a_n \in \mathrm{L}_{\omega_1}$. Prove that

$$\varphi(a_1, \ldots, a_n) \longleftrightarrow \varphi^{\mathrm{L}_{\omega_1}}(a_1, \ldots, a_n).$$

7. Let A and B be r.e. sets so that $A \cap B = \emptyset$.

(a) Show that there is a formula φ in the language of arithmetic so that: (i) If $n \in A$ then $\mathsf{PA} \vdash \varphi(\tilde{n})$; and (ii) If $n \in B$ then $\mathsf{PA} \vdash \neg\varphi(\tilde{n})$. ($\tilde{n}$ here is the formal term in the language of arithmetic for the nth successor to 0.)

(b) Is it possible to strengthen the above to require also: (iii) For every $n \in \omega$, ($\mathsf{PA} \vdash \varphi(\tilde{n})$ or $\mathsf{PA} \vdash \neg\varphi(\tilde{n})$) ?