LOGIC QUALIFYING EXAMINATION

Tuesday, September 21, 2004

Answer all 8 questions. All questions have equal value.

Standard notations:

PA is Peano arithmetic.
ZFC is Zermelo-Fraenkel set theory (with the Axiom of Choice).
For each ordinal λ, V_λ is the set of sets with rank $< \lambda$.
φ_n is the nth recursive partial function, in some standard enumeration.
$W_e = \{x \in \mathbb{N} \mid \varphi_e(x)\downarrow\}$ = the eth recursively enumerable set.
$\ulcorner\theta\urcorner$ is the Gödel number of the formula θ, in some standard Gödel numbering of the relevant formal language. Sentences are "identified" with their Gödel numbers, so that a theory is axiomatizable if the set (of Gödel numbers of) its sentences is recursively enumerable, a function on sentences to sentences is recursive if the corresponding function on Gödel numbers is recursive, etc.

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1. Let $\mathcal{L}$ contain a two-place relation symbol P. Let $\mathfrak{A}$ be an infinite model for $\mathcal{L}$ such that $P_{\mathfrak{A}}$ is an equivalence relation. Prove that at least one of the following holds:

(i) There is some positive integer n such that there are infinitely many n-element equivalence classes of $P_{\mathfrak{A}}$.

(ii) There is an elementary extension $\mathfrak{B}$ of $\mathfrak{A}$ such that $P_{\mathfrak{B}}$ is an equivalence relation with an infinite equivalence class.

2. Let T be a consistent theory in a countable language containing a two place predicate symbol $<$. Assume that $T \vdash$ "$<$ is a linear order of the universe." Assume also that T has no model of size $\aleph_1$ in which each point has only countably many $<$-predecessors. Prove that there is a countable model $\mathfrak{A}$ of T such that, for all $\mathfrak{B} \succ \mathfrak{A}$,

$$(\exists a \in A)(\exists b \in B \setminus A)\, b <_{\mathfrak{B}} a.$$

3. A *sound interpretation* of Peano arithmetic into a theory T (in any language) is a recursive function $\theta \mapsto \theta^*$ on the sentences of PA to the sentences of T which satisfies the following properties, for every sentence θ in the language of PA:

(1) If $PA \vdash \theta$, then $T \vdash \theta^*$.

(2) If $T \vdash \theta^*$, then θ is true.

(3) $\left(\neg\theta\right)^* \equiv \neg\theta^*$.

Prove that if T is axiomatizable and there exists a sound interpretation of PA into T, then T is incomplete.

Hint. Use the Fixed Point Lemma in Peano Arithmetic.

4. Prove that the relation

$$P(e,m) \iff W_e = W_m$$

is Π_2 but not Σ_2.

5. (a) Prove that if $f : \mathbb{N} \to \mathbb{N}$ is a total, recursive function, then there is a number z such that $W_z = \{f(z)\}$.

(b) Prove that there is a number z such that $\varphi_z(z)\downarrow$ and $W_z = \{\varphi_z(z)\}$.

(c) Suppose that $f : \mathbb{N} \to \mathbb{N}$ is a (total) recursive function. Prove that there is a number z such either $f(z) = z$ and $W_z = \{0\}$, or $f(z) \neq z$ and $W_z = \{1\}$.

6. Prove that if κ is a strongly inaccessible cardinal, then there exists a cardinal $\lambda < \kappa$ of cofinality ω such that $V_\lambda \models \mathrm{ZFC}$.

7. Let M be a transitive model of ZFC and let N be a non-transitive elementary submodel of M. Let α be the least ordinal number such that $\alpha \in N$ and $\alpha \not\subseteq N$. Prove that α is a cardinal number in M.

Hint. Let κ be the cardinality of α in the sense of N. Assuming that $\kappa < \alpha$, show that $\alpha \subseteq N$. Conclude that $\kappa = \alpha$. Argue from here that κ is a cardinal in M.

8. Assume that ZFC has a transitive set model. Let α be the least ordinal number such that

$$(L_\alpha; \in) \models \text{ZFC}.$$

Prove that every member of L_α is definable in $(L_\alpha; \in)$. I.e., prove that for every $x \in L_\alpha$ there is a formula $\varphi(v_1)$ such that x is the unique $y \in L_\alpha$ such that

$$(L_\alpha; \in) \models \varphi[y].$$

Hint. Let A be the set of all $y \in L_\alpha$ such that y is definable in $(L_\alpha; \in)$. You may assume without proof that $(A; \in) \prec (L_\alpha; \in)$.