Logic Qualifying Exam **Winter 2004**

Please answer all **eight** questions. **Good luck!**

Question 1. Let T be the theory of the model $(\mathbb{N}; +, \cdot)$. Show that T has uncountably many non-isomorphic countable models.

Question 2. Let $\langle \phi_e \mid e < \omega \rangle$ be a standard enumeration of the recursive partial functions. Show that $\{e \mid \phi_e \text{ is bounded}\}$ is Σ_2 complete. (A partial recursive function is **bounded** if its range is bounded in ω.)

Question 3. Let φ be Goldblach's conjecture: Any even number ≥ 4 is the sum of two primes. Let T be some system of axioms stronger than ZFC, and suppose that $\mathsf{T} \vdash \varphi$. Prove that $\mathrm{CON}(\mathsf{T}) \rightarrow \varphi$.

Question 4. Work with the language of arithmetic and the standard Gödel numbering of sentences. For each sentence σ let $\mathrm{ConSeq}(\sigma)$ be the set of consequences of σ, namely $\{\tau \mid \sigma \vdash \tau\}$. Let $A = \{\ulcorner \sigma \urcorner \mid \mathrm{ConSeq}(\sigma) \text{ is decideable}\}$. Is A recursive? (Prove your answer.)

Question 5. Let $\mathcal{L}$ be the language consisting of the logical symbols and a binary relation symbol R. We view models of $\mathcal{L}$ as graphs, with the universe of the model determining the nodes, and the interpretation of R determining the edges. Is there a theory T in $\mathcal{L}$ such that (for every model $\mathfrak{A}$ of $\mathcal{L}$) $\mathfrak{A} \models T$ iff $\mathfrak{A}$ is *connected* as a graph? (Prove your answer.)

Question 6. Assume the CH (but not the GCH). Prove that ${\omega_n}^{\omega} = \omega_n$ for every natural number $n \geq 1$.

Question 7. Prove that there is a Π_2 sentence φ which is true in L_{ω_1} but false in L.

Question 8. Let $\mathcal{L}$ be the language of arithmetic, and let $\mathfrak{N}$ be the standard model of arithmetic. Let $\langle \theta_n \mid n < \omega \rangle$ be some standard enumeration of the sentences of $\mathcal{L}$.

(a) Recall that the fixed point lemma states that for every formula $\varphi(v)$ of $\mathcal{L}$ there exists a number n so that $\mathfrak{N} \models (\theta_n \leftrightarrow \varphi(\underline{\mathbf{n}}))$. Sketch a proof of this lemma.

(b) Prove that truth is not definable in $\mathfrak{N}$. More precisely prove that there is no formula $\tau(v)$ in $\mathcal{L}$ with the property that $\mathfrak{N} \models \theta_n$ iff $\mathfrak{N} \models \tau(\underline{\mathbf{n}})$.