Qualifying Examination
LOGIC
Fall 2005

All questions have equal value, so try to answer all of them.

You may (and you will need to) use some of the "big" theorems of logic (the Gödel Completeness and Incompleteness Theorems, Tarski's Theorem, Kleene's Normal Form Theorem, the Condensation Lemma, etc.), and when you do, make sure you quote them correctly.

You may also assume that Peano arithmetic (PA) and Zermelo-Fraenkel Set Theory with Choice (ZFC) are consistent.

Question 1. A group $\mathfrak{G} = (G; \cdot, e)$ is *torsion*, if for every $g \in G$ there exists some $n \in \mathbb{N}$ such that $g^n = e$, where e is the identity and $g^n = g \cdot g \cdots g$ (n-times); $\mathfrak{G}$ is of *unbounded rank* if there is no single n such that $g^n = e$ for all $g \in G$.

1a. Prove that every countable torsion group $\mathfrak{G}$ of unbounded rank has a countable, elementary extension which is not torsion.

1b. Prove that there is no first order theory in the language of groups whose models are exactly the torsion groups.

Question 2. Recall that a model is *atomic* if the only [complete] types realized in the model are principal.

Show that if a countable complete theory T has only countably many non-isomorphic countable models, then it has an atomic model.

Question 3. For each natural number n, $\Delta n = S^n(0)$ is the *numeral* which denotes n in the standard (and every) model of Peano Arithmetic, PA.

Prove that there is a formula $\phi(v)$ in the language of PA with just one free variable v, such that

(1) For every $n \in \mathbb{N}$, $\text{PA} \vdash \phi(\Delta n)$.
(2) $\text{PA} \nvdash (\forall v)\phi(v)$.

(Here $\phi(\Delta n)$ is the sentence resulting from $\phi(v)$ by replacing v in all its free occurrences by the numeral Δn.)

Question 4. For each of the following statements, determine whether it is true or false and prove your answer. *Caution*: Two of the statements are reformulations of standard results and one of them is completely trivial.

4a. If $A \subseteq B \subseteq \mathbb{N}$, A is recursively enumerable and B is Π^0_1, then there exists a recursive set C such that $A \subseteq C \subseteq B$.

4b. If $A \subseteq B \subseteq \mathbb{N}$, A is Π^0_1 and B is recursively enumerable, then there exists a recursive set C such that $A \subseteq C \subseteq B$.

4c. If $A \subsetneq B \subseteq \mathbb{N}$ and both A and B are recursively enumerable, then there exists a recursive set C such that $A \subsetneq C \subsetneq B$.

Question 5. Assume that the Kleene (computation) predicate $T_1(e, x, y)$ is defined in some natural way, so that it has the standard properties:

(a) $T_1(e, x, y)$ is (primitive) recursive.
(b) If we set $\varphi_e(x) = U(\mu y T_1(e, x, y))$, then $\varphi_0, \varphi_1, \ldots$ enumerates all the unary, recursive partial functions.
(c) For all e, x,
$$\text{if } T_1(e, x, y), \text{ then } x < y.$$

(The last property (c) holds because the computation which establishes that $\varphi_e(x) = w$ for some w has larger code than the input x.) As usual, we set
$$W_e = \{n \mid \varphi(x)\downarrow\}.$$

A number e is *self-verifying* if for all x, y,
$$\text{if } T_1(e, x, y), \text{ then } y \in W_e.$$
Classify in the arithmetical hierarchy the set
$$A = \{e \mid e \text{ is self-verifying}\}.$$

Question 6. Prove that for each ordinal number α, the structure $(\alpha; \in)$ is *rigid*, i.e., there is no bijection $\pi : \alpha \to \alpha$ (other than the identity) such that
$$\beta \in \gamma \iff \pi(\beta) \in \pi(\gamma) \qquad (\beta, \gamma \in \alpha).$$

Question 7. For this problem, assume that every set is constructible, $V = L$.

Recall that a set of ordinals $A \subseteq \omega_1$ below the first uncountable ordinal is *stationary* if it intersects every closed, unbounded subset of ω_1. Let
$$A = \{\alpha < \omega_1 \mid L_\alpha \models \text{“there exists an uncountable cardinal”}\}.$$

7a. Prove that A is non-empty, and, in fact, unbounded in ω_1.

7b. Prove that A is not stationary.

Question 8. *Caution*: This problem is very easy if you use some basic theorems from Recursion Theory and Proof Theory (in part 8a) and from Proof Theory (in part 8b), but practically impossible to do from scratch.

We assume that the sentences of ZFC and PA have been coded in some standard way, and so that (for convenience), every natural number n is the code of some sentence θ_n^T, with $T = \text{ZFC}$ or $T = \text{PA}$. (You can do this starting from any coding and setting $\theta_n \equiv (\forall x)[x = x]$ if n is not the code of a sentence.) Set
$$T_{\text{ZFC}} = \{n \mid \text{ZFC} \vdash \theta_n^{\text{ZFC}}\}, \qquad T_{\text{PA}} = \{n \mid \text{PA} \vdash \theta_n^{\text{PA}}\}.$$

8a. Prove that there is a recursive permutation $\pi : \mathbb{N} \to \mathbb{N}$ of the natural numbers such that
$$n \in T_{\text{ZFC}} \iff \pi(n) \in T_{\text{PA}}.$$

8b. Show that (the formal version of) **8a** cannot be proved in ZFC.