Logic Qualifying exam **Winter 2005**

Please answer all **eight** questions.

Question 1. Work in ZFC. Prove that there is a limit ordinal α so that (a) α is countable; and (b) there is no bijection $f: \omega \to \alpha$ with $f \in L_{\alpha+1}$.

Question 2. Let $M \subseteq N$ be transitive models of ZFC. Let $r \in M$ be a relation and suppose that (r is wellfounded)^M. Prove that (r is wellfounded)^N.

Question 3. Work in ZF without AC.

(a) A **non-principal filter** over ω is a set $\mathcal{F} \subseteq \mathcal{P}(\omega)$ so that ($\forall n \in \omega$ and $\forall x, y \subseteq \omega$) $\{n\} \notin \mathcal{F}$; $x, y \in \mathcal{F} \to x \cap y \in \mathcal{F}$; and $x \supseteq y \in \mathcal{F} \to x \in \mathcal{F}$. Let $a_0, \ldots, a_k \subseteq \omega$. Prove that there is a non-principal filter $\mathcal{F}$ over ω so that for every $i \leq k$, either $a_k \in \mathcal{F}$ or $\omega - a_k \in \mathcal{F}$.

(b) A non-principal filter $\mathcal{F}$ over ω is called an **ultrafilter** if for every $a \subseteq \omega$ either $a \in \mathcal{F}$ or $\omega - a \in \mathcal{F}$. Assume the compactness theorem: for every language $\mathcal{L}$ and every set Γ of sentences in $\mathcal{L}$, if every finite $\Delta \subseteq \Gamma$ has a model, then so does Γ. Prove that there is a non-principal ultrafilter over ω.

Question 4. Let $\mathfrak{A}$ be a saturated infinite model, and let $a_0, \ldots, a_k \in A$. Prove that there is a (strict) substructure $\mathfrak{B} \subsetneq \mathfrak{A}$ and $\pi: \mathfrak{A} \to \mathfrak{B}$ so that (a) π is an isomorphism; and (b) $\pi(a_0) = a_0, \ldots, \pi(a_k) = a_k$.

Question 5. Let $W_e = \{x \mid \phi_e(x) \downarrow\}$ be the eth recursively enumerable set, in the standard coding. Prove that there is a recursive partial function $f(e)$ such that

$$\text{if } W_e \text{ is infinite, then } f(e)\downarrow,\ f(e) \in W_e, \text{ and } f(e) > 2e.$$

Question 6. **(a)** Prove that there is a number e such that $W_e = \{e\}$.

(b) Prove that Question 5 cannot be strengthened to demand that the required function f be total, i.e.,: there is no total, recursive function f such that

$$\text{if } W_e \text{ is infinite, then } f(e) \in W_e, \text{ and } f(e) > 2e.$$

Question 7. Let $\mathcal{L}$ be the language of set theory. Let T be a consistent extension of ZFC, in the language $\mathcal{L}$. Prove that T is not finitely axiomatizable. Point out a part of the argument where it is important that all the sentences of T are in the language of set theory (as opposed to some larger language $\mathcal{L}^*$).

Question 8. A sentence ϕ in the language of arithmetic is Π_1 if it is of the form

$$\phi \equiv (\forall x_1) \cdots (\forall x_n)\theta$$

where θ has only bounded quantifiers. Let P be Peano arithmetic, and prove that for every Π_1 sentence ϕ,

$$P, \mathrm{Con}_P(\ulcorner \phi \urcorner) \vdash \phi,$$

where $\mathrm{Con}_P(\ulcorner \phi \urcorner)$ expresses in a natural way the consistency of ϕ with Peano arithmetic, in other words it is $\neg \mathrm{Prov}_P(\ulcorner \neg\phi \urcorner)$.