Qualifying Examination
LOGIC
Spring 2006

All questions have equal value, so try to answer all of them.

You may (and you will need to) use some of the "big" theorems of logic (the Gödel Completeness and Incompleteness Theorems, Tarski's Theorem, Kleene's Normal Form Theorem, the Condensation Lemma, etc.), and when you do, make sure you quote them correctly.

You may also assume that Peano arithmetic (PA) is sound (i.e., its theorems are all true in its standard interpretation) and that Zermelo-Fraenkel Set Theory with Choice (ZFC) is consistent.

Question 1. Consider the theory T of an infinite, discrete linear ordering without end points. Show that there is a countable model of T into which every model of T can be elementarily embedded. Conclude that T is complete.

Question 2. Give an example of a countable complete theory with exactly three non-isomorphic countable, infinite models.

Question 3. Suppose A is an infinite, r.e. but not recursive set of natural numbers. You must prove your answers to the following T or F questions. ($|B|$ is the cardinality of the set B.)

3a. True or False: there is a total recursive function $f(x)$ such that for every x,

$$|\{t \in A \mid t \leq f(x)\}| \geq x.$$

3b. True or False: there is an unbounded, total recursive function $f(x)$ such that for every x,

$$|\{t \in A \mid t \leq f(x)\}| = x.$$

Question 4. Assume that ZFC has a well founded model, i.e., there exists a set A and a well-founded relation E on A such that the structure (A, E) is a model of ZFC; prove that ZFC has a least transitive model, i.e., there exists a transitive set M such that $(M, \in \restriction M)$ is a model of ZFC, and M is a subset of every other such transitive model of ZFC.

Question 5. For each set of ordinals A, let $L[A] = \bigcup_\xi L_\xi[A]$, where $L_0[A] = \emptyset$,

$$L_{\eta+1}[A] = \{X \subseteq L_\eta[A] \mid X \text{ is definable (with parameters) in } (L_\eta[A], A \cap L_\eta[A], \in \cap L_\eta[A] \times L_\eta[A])\},$$

and for limit λ, $L_\lambda = \bigcup_{\xi<\lambda} L_\xi[A]$.

Prove that if $A \subseteq \omega_1$ and $X \subseteq \omega$ is a set of finite ordinals, then

$$X \in L[A] \implies (\exists \xi, \eta < \omega_1)[X \in L_\eta[A \cap \xi]].$$

Question 6. For each sentence θ of the language of Peano arithmetic, let $\ulcorner\theta\urcorner$ be its Gödel number, in some canonical way of assigning Gödel numbers to formulas, and let

$$T_{\mathrm{PA}} = \{\ulcorner\theta\urcorner \mid \mathrm{PA} \vdash \theta\},$$
$$R_{\mathrm{PA}} = \{\ulcorner\theta\urcorner \mid \mathrm{PA} \vdash \neg\theta\}.$$

(These are the sets of *provable* and *refutable* sentences of PA.)

6a. Show that T_{PA} and R_{PA} are recursively enumerable.

6b. Show that T_{PA} and R_{PA} are *recursively inseparable*, i.e., there is no recursive set C such that

$$T_{\mathrm{PA}} \subseteq C \text{ and } R_{\mathrm{PA}} \cap C = \emptyset.$$

Hint: You may find it easier to first prove that there exists a pair of r.e. recursively inseparable sets of natural numbers, and then use a basic result about the representability of r.e. sets in PA.

Question 7. Let $\Delta(n)$ be the *numeral* of n, i.e., some canonical (closed) formal term of PA which denotes the number n; let $\mathrm{Con}_{\mathrm{PA}}$ be the formal sentence of PA which asserts the consistency of PA; and let $\mathrm{Prov}(v)$ be a formula of PA which defines (in the canonical way) the relation "*v is the Gödel number of a provable sentence*", so that for each sentence θ

$$\mathrm{PA} \models \mathrm{Prov}(\Delta(\ulcorner\theta\urcorner)) \iff \mathrm{PA} \vdash \theta.$$

Determine which of the following implications are provable in PA, and prove your answers. (You may refer to any standard theorems about Peano Arithmetic, after you state them correctly.)

(a) $\mathrm{Con}_{\mathrm{PA}} \rightarrow \mathrm{Prov}(\Delta(\ulcorner\mathrm{Con}_{\mathrm{PA}}\urcorner))$.

(b) $\mathrm{Con}_{\mathrm{PA}} \rightarrow \neg\mathrm{Prov}(\Delta(\ulcorner\mathrm{Con}_{\mathrm{PA}}\urcorner))$.

(c) $\mathrm{Prov}(\Delta(\ulcorner\mathrm{Con}_{\mathrm{PA}}\urcorner)) \rightarrow \mathrm{Con}_{\mathrm{PA}}$.

(d) $\neg\mathrm{Prov}(\Delta(\ulcorner\mathrm{Con}_{\mathrm{PA}}\urcorner)) \rightarrow \mathrm{Con}_{\mathrm{PA}}$.

Question 8. Let $\varphi_e(x)$ be the recursive partial function with code e and

$$W_e = \{x \mid \varphi_e(x)\downarrow\},$$

as usual, and suppose that $f(z)$ is a recursive partial function such that

$$W_e = W_m \implies f(e) = f(m).$$

Prove that if W_e is infinite and $f(e)\downarrow$, then there is a finite $W_m \subseteq W_e$ such that $f(e) = f(m)$. *Hint.* Given W_e, apply the Second Recursion Theorem to the recursive partial function

$$g(m, x) = \begin{cases} \varphi_e(x), & \text{if } (\forall y \leq x)\neg[T_1(\hat{f}, m, y) \,\&\, U(y) = f(e)] \\ \uparrow \text{(diverges)}, & \text{otherwise,} \end{cases}$$

where $f(z) = \varphi_{\hat{f}}(z) = U(\mu y T_1(\hat{f}, z, y))$.