Qualifying Examination
LOGIC
Fall 2007

All questions have equal value, so try to answer all of them.

In the questions which have several parts, the first parts are easier and can be used to answer the more difficult ones which follow; so do the first parts first.

You may (and you will need to) use some of the "big" theorems of logic (the Gödel Completeness and Incompleteness Theorems, Tarski's Theorem, Kleene's Normal Form Theorem, the Condensation Lemma, etc.), and when you do, make sure you quote them correctly.

You may also assume that Peano arithmetic (PA) is sound (i.e., its theorems are all true in its standard interpretation) and that Zermelo-Fraenkel Set Theory with Choice (ZFC) is consistent.

Question 1.

(1a) Prove that there is an end extension of $(\mathbb{N}; 0, S, +, \cdot)$ which satisfies Peano arithmetic but is not an elementary extension of $(\mathbb{N}; 0, S, +, \cdot)$.

(1b) Let $\mathfrak{A}$ be an elementary end extension of $(\mathbb{N}; 0, S, +, \cdot)$. Prove that there is an end extension of $\mathfrak{A}$ which satisfies Peano Arithmetic but is not an elementary extension of $\mathfrak{A}$.

Question 2. Work in the constructible universe L. For each $\alpha < \omega_1$ let $\beta(\alpha) \geq \alpha$ be least so that $L_{\beta(\alpha)+1}$ has a surjection of ω onto α.

Let $S \subseteq \omega_1$. Let φ be a Σ_1 formula so that $L \models \varphi[S]$. Prove the following:

(2a) There is a transitive set model N so that $N \models \varphi[S]$.

(2b) There is a transitive countable model $\bar{N}$, and an ordinal $\alpha \in \bar{N}$ so that $\bar{N} \models \varphi[S \cap \alpha]$.

(2c) There is an $\alpha < \omega_1$ so that $L_{\beta(\alpha)} \models \varphi[S \cap \alpha]$.

Question 3. Let $\pi \colon N \to V_\theta$ be elementary and non-trivial, where θ is some cardinal, and N is transitive. Let $\kappa = crit(\pi)$, that is the least ordinal so that $\pi(\kappa) \neq \kappa$. Prove that $N \models$"κ is a regular cardinal."

Question 4. Let $\mathcal{L}$ be the language consisting of one binary relation symbol P. A model $\mathfrak{A}$ in the language $\mathcal{L}$ is a *graph*: the set of nodes is the universe of $\mathfrak{A}$, and two nodes x, y are connected by a vertex iff $\mathfrak{A} \models P[x, y]$. Prove that there is no sentence ψ in $\mathcal{L}$ so that $\mathfrak{A} \models \psi$ iff $\mathfrak{A}$ is two colorable.

Question 5. Let $\mathfrak{A} = (\mathbb{R}; 0, S, +, \cdot)$ and let T be the theory of $\mathfrak{A}$. A real x is *specifiable* if there is a formula φ so that $\varphi[x]$ holds in $\mathfrak{A}$ and $\varphi[y]$ fails in $\mathfrak{A}$ for all $y \neq x$. Prove that the following are equivalent:

- There is $B \subseteq \mathbb{R}$ so that $(B; 0, S, +, \cdot) \models T$ and $x \notin B$.
- x is not specifiable.

Question 6. Suppose $R(\vec{x}, w)$ is an r.e. relation such that for each $\vec{x}$ there is at least one w such that $R(\vec{x}, w)$.

(6a) Prove that there is a total recursive function $g(\vec{x})$ such that $(\forall \vec{x})R(\vec{x}, g(\vec{x}))$.

(6b) Prove that there is a total recursive function $f(n, \vec{x})$ such that

$$R(\vec{x}, w) \iff (\exists n)[w = f(n, \vec{x})], \tag{1}$$

i.e., for each $\vec{x}$,

$$\{w \mid R(\vec{x}, w)\} = \{f(0, \vec{x}), f(1, \vec{x}), \ldots\}.$$

(6c) Prove that if (in addition), for each $\vec{x}$ there are infinitely many w such that $R(\vec{x}, w)$, then there is a total recursive function $f(n, \vec{x})$ which satisfies (1) and is "1–1 in n", i.e., for all $\vec{x}, m, n$,

$$m \neq n \implies f(m, \vec{x}) \neq f(n, \vec{x}).$$

Question 7. We let $\ulcorner \theta \urcorner$ be the Gödel number of a sentence θ in the language of PA (Peano arithmetic), in some standard Gödel numbering; we let $\Delta(n)$ be the *numeral* of n, i.e., some canonical (closed) formal term of PA which denotes the number n; and we let $\text{Prov}(v)$ be a formula of PA which defines (in the canonical way) the relation "*v is the Gödel number of a provable sentence*".

A formula θ in the language of PA (Peano arithmetic) is Σ_1 if it is of the form

$$\theta \equiv (\exists x_1)(\exists x_2) \cdots (\exists x_n)\psi$$

where ψ has only bounded quantifiers. (For example, $\text{Prov}(v)$ is a Σ_1 formula, as it asserts that "there exists a proof ...".)

(7a) Consider the following four sentences of PA constructed from an arbitrary Σ_1-sentence θ and determines which of them are **true**. You will get half-credit for each of the four for which you give the correct answer, and full credit for each of them for which you justify your answer.

(1) $\text{Prov}(\Delta(\ulcorner \theta \urcorner)) \to \theta$.
(2) $\theta \to \text{Prov}(\Delta(\ulcorner \theta \urcorner))$.
(3) $\text{Prov}(\Delta(\ulcorner \neg\theta \urcorner)) \to \neg\theta$.
(4) $\neg\theta \to \text{Prov}(\Delta(\ulcorner \neg\theta \urcorner))$.

(7b) For each of (1) – (4) constructed from an arbitrary Σ_1-sentence, determine whether it is **provable** in PA. (Same rules for the credit.)

Question 8. Let $\langle \varphi_e \mid e \in \omega \rangle$ be the standard enumeration of the recursive functions from ω to ω, and let $W_e = \{n \mid \varphi_e(n) \downarrow\}$ as usual, so that $\langle W_e \mid e \in \omega \rangle$ is the standard enumeration of the r.e. subsets of ω. Let V be the set of even numbers. Prove that $\{e \mid V \subseteq W_e\}$ is Π_2 complete.