Qualifying Examination
LOGIC
Fall 2008

All questions have equal value, so try to answer the easier ones first.

You may (and you will need to) use some of the "big" theorems of logic (the Gödel Completeness and Incompleteness Theorems, Tarski's Theorem, Kleene's Normal Form Theorem, the Condensation Lemma, etc.), and when you do, make sure you quote them correctly.

You may also assume that Peano arithmetic (PA) is sound (i.e., its theorems are all true in its standard interpretation) and that Zermelo-Fraenkel Set Theory with Choice (ZFC) is consistent.

Problem 1. An *ordering* is a structure $(A, \leq)$ with one binary relation $\leq$ which is a linear ordering of the universe A, and it is a *wellordering* if $\leq$ is well founded, i.e., if there are no infinite, $>$-descending chains. Put

$$\begin{aligned} \mathcal{W} &= \{(A, \leq) \mid (A, \leq) \text{ is a wellordering } \}, \\ \mathcal{W}^c &= \{(A, \leq) \mid \text{ is an ordering but not a wellordering}\}. \end{aligned}$$

(1a) Determine whether $\mathcal{W}$ is an elementary class or not and prove your answer.

(1b) Determine whether $\mathcal{W}^c$ is an elementary class or not and prove your answer.

Problem 2. Recall that $W_e = \{x \mid \varphi_e(x)\downarrow\}$, where $\varphi_e(x) = U(\mu y T_1(e, x, y))$ is the recursive partial function with code e (in some standard enumeration).

Let $E = \{0, 2, 4, \ldots\}$ be the set of even numbers. Classify in the arithmetical hierarchy the following two sets:

(2a) $A = \{e \mid W_e \subseteq E\}$.

(2b) $B = \{e \mid E \subseteq W_e\}$.

Problem 3. Consider the structure

$$\mathcal{R} = (\mathbb{R}, 0, 1, <, f),$$

where $(\mathbb{R}, 0, 1, <)$ is the set of real numbers with the usual ordering and the distinguished elements $0, 1$, and $f(x) = 2x$. Prove that $\mathcal{R}$ admits effective quantifier elimination, and that its elementary theory is decidable.

Problem 4. Suppose $g : \mathbb{N} \rightharpoonup \mathbb{N}$ is a recursive partial function such that for all e,

$$W_e = \emptyset \implies g(e)\downarrow .$$

Prove that there is some m such that

$$W_m = \{m\} \text{ and } g(m)\downarrow .$$

Problem 5. For each sentence θ in the language of PA (Peano arithmetic), let $\Box\theta$ be the sentence which expresses formally the provability of θ, i.e.,

$$\Box\theta :\equiv (\exists y)\mathbf{Proof}_{\mathrm{PA}}(\ulcorner\theta\urcorner, y),$$

where $\ulcorner\theta\urcorner$ is the numeral of the code (Gödel number) of θ and $\mathbf{Proof}_{\mathrm{PA}}(\mathsf{v}_1, \mathsf{v}_2)$ numeralwise expresses the relation

$$\mathrm{Proof}_{\mathrm{PA}}(e, y) \iff e \text{ is the code of a sentence } \theta \text{ and } y \text{ is the code of a proof of } \theta \text{ in PA.}$$

For each of the following four sentence schemes, determine whether or not it is true for all choices of sentences ϕ, ψ, and prove your answers:

(a) $$(\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \Box\psi)$$

(b) $$(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \psi)$$

(c) $$(\phi \rightarrow \Box\psi) \rightarrow (\Box\phi \rightarrow \psi)$$

(d) $$(\Box\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \Box\psi)$$

Problem 6. For each of the sentence schemes (a) – (d) in Problem 5, determine whether or not it is provable in PA for all choices of sentences ϕ, ψ, and prove your answers.

Problem 7. Let $M \subseteq V$ be a transitive class model of ZFC. Let $\tau < \kappa \in M$ be cardinals of V.

(7a) Suppose $(\kappa \text{ is regular})^V$. Prove $(\kappa \text{ is regular})^M$.

(7b) Prove that (r is a wellordering of τ) is absolute between M and V.

(7c) Suppose r is a wellordering of τ. Let δ be the ordertype of r. Suppose $r \in M$. Prove that $(|\delta| = \tau)^M$.

(7d) Suppose $(\delta^+ = \kappa)^L$ and $(|\delta| = \tau)^M$. Prove that $(\tau^+ = \kappa)^M$.

Note: Some of the parts of this problem can be done independently of those preceding them.

Problem 8. Work in ZFC, and assume that for every $\delta < \omega_1$, $(\delta^+)^L < \omega_1$; prove Con(ZFC).

Hint. Let κ denote $\omega_1 = (\omega_1)^V$ and consider the model L_κ.
