UCLA Department of Mathematics

Qualifying Examination

LOGIC

Fall 2009

All questions have equal value, so try to answer the easier-looking ones first.

In the questions which have several parts, the first parts are easier and can often be used to answer the more difficult ones which follow; so do the first parts first.

You may (and you will need to) use some of the "big" theorems of logic (the Gödel Completeness and Incompleteness Theorems, Tarski's Theorem, etc.), and when you do, make sure you quote them correctly.

You may assume that Peano Arithmetic PA is sound (i.e., it theorems are all true in its standard interpretation) and that Zermelo-Fraenkel Set Theory with Choice (ZFC) is consistent.

(1) Let T be an axiomatizable extension of PA (in the language of PA), suppose $\mathbf{Proof}_T(v, u)$ numeralwise expresses in PA the relation

$$\mathrm{Proof}_T(x, y) \iff x \text{ is the code of PA-sentence } \theta$$
$$\text{and } y \text{ is the code of a proof of } \theta \text{ in } T,$$

and set
$$\mathbf{Provable}_T(v) :\equiv (\exists u)\mathbf{Proof}_T(v, u).$$

Write $\ulcorner \sigma \urcorner$ for the numeral of the Gödel number of the sentence σ, in some standard Gödel number; let $\Delta(n)$ be the numeral of n.

(a) True or false: for any PA-sentence σ,

$$\text{PA} \vdash \mathbf{Provable}_{T \cup \{\neg\sigma\}}(\ulcorner \sigma \urcorner) \rightarrow \mathbf{Provable}_T(\ulcorner \sigma \urcorner).$$

For a fixed formula $\theta(v)$ with just v free, let

$$R_\theta(x, y) \iff y \text{ is the code of the term } \ulcorner \theta(\Delta x) \urcorner,$$

and let $\mathbf{R}_\theta(v, w)$ numeralwise express R_θ in PA.

(b) True or false, for any formula $\theta(v)$ with just v free:

$$\text{PA} \vdash \mathbf{Provable}_T(\ulcorner (\forall v)\theta(v) \urcorner) \rightarrow (\forall v)(\exists w)[\mathbf{R}_\theta(v, w) \,\&\, \mathbf{Provable}_T(w)].$$

(c) True or false, for θ and $\mathbf{R}_\theta$ as above:

$$\text{PA} \vdash (\forall v)(\exists w)[\mathbf{R}_\theta(v, w) \,\&\, \mathbf{Provable}_T(w)] \rightarrow (\forall v)\theta(v).$$

(2) True or false? For every two structures $\mathcal{A}$ and $\mathcal{B}$ for the same language, there is a sentence σ which is valid in both $\mathcal{A}$ and $\mathcal{B}$ but not logically valid.

(3) Let $\mathcal{L}$ be a finite language and T a finite consistent $\mathcal{L}$-theory. Suppose that any two countable models of T are isomorphic. Show that T is decidable.

(4) Let $\mathcal{L} = \{<\}$ with a binary relation symbol $<$.
  (a) Let $\mathcal{M} = (M, <^{\mathcal{M}})$ be an ordered set. We say that $\mathcal{M}$ can be extended to a total ordering if there is a total ordering $\prec$ of M with $<^{\mathcal{M}} \subseteq \prec$. Show that $\mathcal{M}$ can be extended to a total ordering if every finite $\mathcal{L}$-substructure of $\mathcal{M}$ can be extended to a total ordering.
  (b) Let $\mathcal{L}^*$ be a language extending $\mathcal{L}$, and let T be a consistent $\mathcal{L}^*$-theory. Suppose that for every $\mathcal{M} \models T$, the interpretation $<^{\mathcal{M}}$ of $<$ in $\mathcal{M}$ is a well-ordering of M, that is, there are no infinite sequences $a_0 >^{\mathcal{M}} a_1 >^{\mathcal{M}} \cdots$. Show that every model of T is finite.

(5) A topology on the class of ordinal numbers is defined by stipulating that
$$\{\{0\}\} \cup \{(\alpha, \beta) : \alpha + 1 < \beta\}$$
is the class of basic open sets, where
$$(\alpha, \beta) = \{\gamma : \alpha < \gamma < \beta\}.$$
The topology of any ordinal is the relative topology. Determine (with proof) the ordinals which are compact in this topology.

(6) Is there an ordinal number $\alpha < \omega_\omega$ such that $L_\alpha \models$ "ω_ω exists"?

(7) Let κ be inaccessible. Let M be an elementary submodel of V_κ such that $\kappa \subseteq M$. Prove that $M = V_\kappa$.

(8) Prove that
$$K = \{e : e \subset W_e\}$$
is one-one reducible to
$$\mathrm{Fin} = \{e : W_e \text{ is finite}\}.$$