Qualifying Examination
LOGIC
Fall 2010

Try to answer all questions.

You may (and you will need to) use some of the "big" theorems of logic (the Gödel Completeness and Incompleteness Theorems, the Omitting Types Theorems, Tarski's Theorem, Kleene's Normal Form Theorem, the Condensation Lemma, etc.), and when you do, make sure you quote them correctly.

You may also assume that Peano arithmetic (PA) is sound (i.e., its theorems are all true in its standard interpretation) and that Zermelo-Fraenkel Set Theory with Choice (ZFC) is consistent.

Note: Some problems or parts of problems are harder than others, or require using methods from more than one part of the subject for their solution. If you cannot quickly solve a problem or part of it, move onward and return to it later.

Problem 1. Let $\ulcorner\theta\urcorner$ be the Gödel number of the sentence θ in the language of PA, and for each e let

$$T_e = \{\theta \mid \ulcorner\theta\urcorner \in W_e\}.$$

We think of each T_e as the set of axioms of a theory in the language of PA. (But T_e need not extend PA, and indeed may axiomatize the symbols of PA in ways completely unrelated to their standard interpretations.)

For each of the following sets, determine whether it is arithmetical and, if it is, classify it in the arithmetical hierarchy (i.e., specify some Σ_n or Π_n and prove that the set in question belongs to one of these classes and not the other).

(1a) $A = \{e \mid T_e \text{ is consistent}\} = \{e \mid T_e \nvdash 0 = 1\}$.

(1b) $B = \{e \mid T_e \text{ is sound}\}$
$= \{e \mid \text{each } \theta \in T_e \text{ is true in the standard model}\}$.

(1c) $C = \{e \mid T_e \text{ is consistent and complete}\}$
$= \{e \mid e \in A \text{ and for all } \theta, T_e \vdash \theta \text{ or } T_e \vdash \neg\theta\}$.

Problem 2. Refer to the definition of T_e in Problem 1.

(2a) Prove that there is a sequence of sentences $\theta_0, \theta_1, \ldots$ such that:

(i) The function $f(e) = \ulcorner\theta_e\urcorner$ is recursive.
(ii) If T_e is a consistent extension of PA, then $T_e \nvdash \theta_e$ and $T_e \nvdash \neg\theta_e$.

(2b) Prove that there is a number e such that

$$T_e = \mathrm{PA} \cup \{\theta_e\}.$$

(2c) Determine which of the following is true for the e in the previous part, and prove your answer:

(i) $\mathrm{PA} \vdash \theta_e$, (ii) $\mathrm{PA} \vdash \neg\theta_e$, (iii) θ_e is not decided by PA.

Problem 3. Let T be a consistent theory in a language whose only non-logical symbol is a three-place relation symbol. If $\mathfrak{A} = (A, R)$ is a model of T and $a \in A$, let

$$R_a = \{(b, c) \mid R(a, b, c)\}.$$

Prove that T has a model $\mathfrak{A}$ such that no R_a is an infinite wellordering.

Problem 4. A model is *atomic* if it realizes no non-principal complete type of its theory. Let T be a complete theory in a countable language with no atomic models. Let $\mathfrak{A}$ be a countable model of T. Prove that there is a countable model $\mathfrak{B}$ of T such that neither of $\mathfrak{A}$ nor $\mathfrak{B}$ can be elementarily embedded into the other.

Hint. Use an omitting types argument.

Problem 5. Let A be a set, and let α be an ordinal. Let $\langle B_\gamma \mid \gamma < \alpha\rangle$ be a sequence of subsets of ${}^\alpha A$. Suppose that $\operatorname{card}(B_\gamma) < \operatorname{card}(A)$ for each $\gamma < \alpha$. Prove that $\bigcup_{\gamma<\alpha} B_\gamma \neq {}^\alpha A$. In other words show that there is an $f \in {}^\alpha A$ so that

$$(\forall \gamma < \alpha)(\forall g \in B_\gamma)\, g \neq f.$$

Problem 6. Let $\sigma_0, \sigma_1, \ldots$ be an (effective) enumeration of the axioms of ZFC, let

$$\mathrm{ZFC}_n = \{\sigma_0, \ldots, \sigma_n\},$$

and let

$$\mathrm{Consis}(n) \iff \text{the theory } \mathrm{ZFC}_n \text{ is consistent.}$$

Let $\mathbf{Consis}(v)$ be a ZFC-formula which (naturally) defines this relation on ω, and for each $n \in \omega$, let $\mathbf{N}_n(v)$ be a ZFC-formula which (naturally) expresses the relation

$$N_n(v) \iff v = n.$$

(6a) Prove that for every number n, $\mathrm{ZFC} \vdash (\exists v)[\mathbf{N}_n(v)\ \&\ \mathbf{Consis}(v)]$.

(6b) Prove that $\mathrm{ZFC} \nvdash (\forall v)[v \in \omega \rightarrow \mathbf{Consis}(v)]$.

(6c) True or false: for every n and every sentence θ (and with the appropriate formal definitions),

$$\mathrm{ZFC} \vdash \Big(\mathrm{ZFC}_n \vdash \theta\Big) \rightarrow \theta.$$

Problem 7. Let $A \subseteq \omega$. Define models $L_\alpha(A)$ as follows: $L_n(A) = L_n$ for $n < \omega$; $L_\omega(A) = L_\omega \cup \{A\}$; for $\alpha \geq \omega$, $L_{\alpha+1}(A)$ consists of all subsets of $L_\alpha(A)$ that are definable with parameters over $L_\alpha(A)$; and $L_\lambda(A) = \bigcup_{\alpha<\lambda} L_\alpha(A)$ for limit $\lambda > \omega$. Define $L(A) = \bigcup_{\alpha \in \mathrm{ON}} L_\alpha(A)$.

Assume that M is a transitive set model of ZFC and that A belongs to M. Suppose that $M \models L(A) \neq L$.

(7a) Does it follow that $M \models A \notin L$?

(7b) Does it follow that $A \notin L$?

(7c) Does it follow that $A \notin L$ if M is uncountable?

(7d) Does it follow that $A \notin L$ if M is countable?

Prove your answers.

Problem 8. A (total) recursive function is *provably recursive* in PA, if PA can prove that it is total, i.e., if $f = \varphi_e$ for some e such that

$$\mathrm{PA} \vdash (\forall x)(\exists y)\mathrm{T}_1(\mathbf{e}, x, y),$$

where $\mathrm{T}_1(u, x, y)$ numeralwise expresses the T predicate and $\mathbf{e} = S^e(0)$ is the canonical PA-term which denotes e.

(8a) Prove that there is a total recursive function which is not provably recursive in PA.

(8b) Describe (informally) the appropriate definition and prove that there is a total recursive function which is provably recursive in ZFC but not provably recursive in PA.

Note. What is required in the second part is to pinpoint the property of ZFC which makes it true, rather than worry about the details of formal definitions or proofs.
