Qualifying Examination
LOGIC
Fall 2011

Try to answer all 8 questions.

You should work in ZFC. Thus, in particular, you may assume that Peano arithmetic (PA) is sound (i.e., its theorems are all true in its standard interpretation).

Note: Some problems or parts of problems are harder than others. If you cannot quickly solve a problem or part of it, move onward and return to it later.

Problem 1. Construe $\mathbb{Z}$ as a first-order structure in the language $\{0, +, -\}$ of abelian groups.

(1a) Determine all elementary substructures of $\mathbb{Z}$.

(1b) Which elements of $\mathbb{Z}$ are definable (without parameters) in $\mathbb{Z}$?

(1c) Are $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$ elementarily equivalent?

(1d) For every prime number p let $\mathbb{Z}_{(p)}$ be the subgroup of $(\mathbb{Q}, 0, +, -)$ consisting of all rational numbers of the form a/b where $a, b \in \mathbb{Z}$, $b \neq 0$, with b not divisible by p. Let $Z := \prod_p \mathbb{Z}_{(p)}$, where the product is finite support (i.e., for each of its elements, all but finitely many coordinates are 0). Is there an elementary embedding $\mathbb{Z} \to Z$?

Problem 2. Let α be a bijection between the set of finite sequences of 0s and 1s and $\mathbb{N}$, and for $f\colon \mathbb{N} \to \{0,1\}$ let $f^*\colon \mathbb{N} \to \mathbb{N}$ be given by $f^*(n) = \alpha(f \restriction \{0, \dots, n-1\})$.

(2a) Show: $(f^*)_{f\colon \mathbb{N}\to\{0,1\}}$ is a continuum size family of almost disjoint functions, i.e., for $f \neq g$ there are only finitely many n such that $f^*(n) = g^*(n)$.

(2b) Show that every proper elementary extension of the first-order structure $(\mathbb{N}, (f^*)_{f\colon \mathbb{N}\to\{0,1\}})$ is uncountable.

Problem 3. Let $\mathcal{L}$, $\mathcal{L}'$ be first-order languages with $\mathcal{L} \subseteq \mathcal{L}'$, let T' be an $\mathcal{L}'$-theory, and let $\mathbf{C}$ be the class of all $\mathcal{L}$-structures which can be expanded to a model of T'. Show: if $\mathbf{C}$ is closed under taking substructures, then $\mathbf{C}$ is universally axiomatizable. (Hint: consider the theory T consisting of all universal $\mathcal{L}$-formulas φ with $T' \models \varphi$.)

Problem 4. Prove that the following are equivalent for subsets A of $\mathbb{N}$.

(1) A is Δ^0_2.
(2) A is recursive in K, where $K = \{e \mid e \in W_e\}$.

(3) There is a recursive function $f : \mathbb{N} \times \mathbb{N} \to \{0, 1\}$ such that for each n $\lim_s f(s, n)$ exists and $n \in A \leftrightarrow \lim_s f(s, n) = 1$.

Problem 5.

(5a) Does every recursive partial function have a recursive total extension? Prove your answer.

(5b) Classify in the arithmetical hierarchy the set of all numbers e such that φ_e has a primitive recursive extension. Here φ_e is the eth partial recursive function in some standard enumeration.

Problem 6. In some standard Gödel numbering of the language of PA, let $\text{Prov}(v)$ be a formula of PA that defines (in the canonical way) the property "v is the Gödel number of a sentence provable in PA." Let Con(PA) be the sentence $\neg\text{Prov}(\mathbf{n})$, where $\mathbf{n}$ is the numeral of the Gödel number of $\mathbf{1} = \mathbf{0}$. Loeb's Theorem for PA states that, for any sentence σ, if

$$\text{PA} \vdash (\text{Prov}(\sigma) \to \sigma),$$

then $\text{PA} \vdash \sigma$.

(6a) Sketch a proof of Loeb's Theorem from the 2nd Incompleteness Theorem.

(6b) Deduce from Loeb's Theorem that $\text{PA} \nvdash \text{Con(PA)}$.

Problem 7. Assume that (M, R) is a well-founded model of ZFC.

(7a) Prove that $(M, R) \models$ "There is a model of ZFC."

(7b) Must (M, R) satisfy "There is well-founded model of ZFC," i.e., does our assumption imply that (M, R) satisfies this sentence?

(7c) Must (M, R) have a proper substructure $(N, R \restriction N)$ such that $(N, R \restriction N) \models$ ZFC and N is transitive relative to (M, R) (i.e., such that if $b \in N$ and $R(a, b)$ then $a \in N$)? It might help to think about the special case that M is a transitive set and R is $\in \restriction M$, when "transitive relative to (M, R)" just means "transitive."

Problem 8. To recall some definitions, let α be a limit ordinal. The *cofinality* of α, $\mathrm{cf}(\alpha)$, is the least cardinal λ such that there is an $f : \lambda \to \alpha$ whose range is unbounded in α. The ordinal α is *regular* if $\mathrm{cf}(\alpha) = \alpha$. A subset of α is a *club in* α if it is unbounded in α and contains the supremum of each of its subsets that is bounded in α. A subset of α is *stationary in* α if it meets every club in α. You may use Fodor's Theorem, which says that if $g : \alpha \to \alpha$ and $\{\beta < \alpha \mid g(\beta) < \beta\}$ is stationary in α, then g is constant on a set stationary in α.

Let κ be an uncountable regular cardinal, and let S be the set of all limit ordinals $\alpha < \kappa$ such that that $\mathrm{cf}(\alpha) = \omega$. For each $\alpha \in S$, let $f_\alpha : \omega \to \alpha$ have unbounded range.

(8a) Prove that there is an $n \in \omega$ such that, for each $\beta < \kappa$, $\{\alpha \in S \mid f_\alpha(n) \geq \beta\}$ is stationary in κ.

(8b) Prove that S is the union of κ disjoint sets stationary in κ.
