Qualifying Examination
LOGIC
Spring 2011

Try to answer all questions.

You may (and you will need to) use some of the "big" theorems of logic (the Gödel Completeness and Incompleteness Theorems, the Omitting Types Theorems, Tarski's Theorem, Kleene's Normal Form Theorem, the Condensation Lemma, etc.), and when you do, make sure you quote them correctly.

You may also assume that Peano arithmetic (PA) is sound (i.e., its theorems are all true in its standard interpretation) and that Zermelo-Fraenkel Set Theory with Choice (ZFC) is consistent.

Note: Some problems or parts of problems are harder than others, or require using methods from more than one part of the subject for their solution. If you cannot quickly solve a problem or part of it, move onward and return to it later.

Problem 1. Let $\varphi_{\exp}(x, y, w)$ be a formula in the signature $\tau = (0, S, +, \cdot)$ of the language of Peano arithmetic which naturally defines the exponential function, so that in particular,

$$k^m = n \iff (\mathbb{N}, 0, S, +, \cdot) \models \varphi_{\exp}[k, m, n], \quad \mathsf{PA} \vdash (\forall x, y)(\exists! w)\varphi_{\exp}(x, y, w).$$

Let exp be a fresh, binary function symbol. Let $\mathsf{PA}^{\exp}$ be the theory in the expanded signature $(0, S, +, \cdot, \exp)$ whose axioms are all the axiom of PA, all instances

$$\Big[\psi(0, \vec{y})\,\&(\forall x)[\psi(x, \vec{y}) \rightarrow \psi(S(x), \vec{y})]\Big] \rightarrow (\forall x)\psi(x, \vec{y}),$$

of the Induction Axiom Scheme where $\psi(x, \vec{y})$ is any $(0, S, +, \cdot, \exp)$-formula, and the characteristic axiom

$$(\forall x)(\forall y)\varphi_{\exp}(x, y, \exp(x, y)).$$

Prove (in outline) that $\mathsf{PA}^{\exp}$ is a *conservative extension of* PA: i.e., for every $(0, S, +, \cdot,)$-sentence θ,

$$\mathsf{PA} \vdash \theta \iff \mathsf{PA}^{\exp} \vdash \theta.$$

Problem 2. Let $\mathcal{L}$ be a first-order language, $\mathbf{N}$ an $\mathcal{L}$-structure and $\mathbf{M}$ a substructure of $\mathbf{N}$.

(2a) Suppose that for all finite subsets A of M and every $c \in N$ there is an automorphism σ of $\mathbf{N}$ with $\sigma(a) = a$ for all $a \in A$ and $\sigma(c) \in M$. Show that $\mathbf{M}$ is an elementary substructure of $\mathbf{N}$.

(2b) Show by an example that the criterion for $\mathbf{M} \preceq \mathbf{N}$ in **(2a)**, although sufficient, is not necessary.

Problem 3. Let $\mathcal{L} = \{P\}$ be a first-order language where P is a unary relation symbol, and

$$T := \left\{ \exists x_1 \cdots \exists x_n \left(\bigwedge_{1 \le i < j \le n} \neg x_i = x_j \wedge \bigwedge_{1 \le i \le n} P x_i \right) : n \in \mathbb{N} \right\},$$

$$T' := \left\{ \exists x_1 \cdots \exists x_n \left(\bigwedge_{1 \le i < j \le n} \neg x_i = x_j \wedge \bigwedge_{1 \le i \le n} \neg P x_i \right) : n \in \mathbb{N} \right\}.$$

(3a) Prove that T and T' are both incomplete.

(3b) Prove that the union $T \cup T'$ is complete.

Problem 4. As usual, $\varphi_e : \mathbb{N} \rightharpoonup \mathbb{N}$ is the (unary) recursive partial function with code (Gödel number) e.

(4a) Prove that there is some number e such that for all x, $\varphi_e(x) = e$.

(4b) Classify in the arithmetical hierarchy the set

$$A = \{e : (\forall x)[\varphi_e(x) = e]\}.$$

Problem 5. Fix natural formalizations (in PA) of the proof relations of PA and ZFC and set

$$\Box_{\mathsf{PA}}\theta \quad :\equiv \quad \text{“}\theta \text{ is a theorem of PA”},$$
$$\Box_{\mathsf{ZFC}}\psi \quad :\equiv \quad \text{“}\psi \text{ is a theorem of ZFC”}.$$

Both $\Box_{\mathsf{PA}}\theta$ and $\Box_{\mathsf{ZFC}}\psi$ are sentences of PA.

For each PA-sentence θ, let

$$\theta^* \equiv \text{the natural translation of } \theta \text{ in the language of ZFC.}$$

(5a) For each of the following four provability claims, either prove that it is true for every PA-sentence θ or provide a counterexample.

(a) $\mathsf{PA} \vdash \Box_{\mathsf{PA}}\theta \to \Box_{\mathsf{ZFC}}\theta^*$.
(b) $\mathsf{PA} \vdash \Box_{\mathsf{ZFC}}\theta^* \to \Box_{\mathsf{PA}}\theta$.
(c) $\mathsf{ZFC} \vdash \left(\Box_{\mathsf{ZFC}}\theta^* \to \theta\right)^*$.
(d) $\mathsf{ZFC} \vdash \left(\Box_{\mathsf{PA}}\theta \to \theta\right)^*$.

(5b) Let PA^* be the extension of PA by the strong *Reflection Principle*

$$\mathsf{PA}^* = \mathsf{PA} \cup \left\{ \Box_{\mathsf{ZFC}}\theta^* \to \theta : \theta \text{ is a PA-sentence} \right\}.$$

(a) True or false: $\mathsf{ZFC} \vdash \mathsf{Consis}(\mathsf{PA})^*$. Prove you answer.
(b) True or false: $\mathsf{PA}^* \vdash \mathsf{Consis}(\mathsf{ZFC})$. Prove you answer.

Problem 6. Recall that a (binary) relation $R \subseteq A \times A$ on a set A is (strictly) well-founded if

$$(1) \qquad \emptyset \neq X \subseteq A \implies (\exists x \in X)(\forall y \in X)\neg R(y, x).$$

A wellordering of a set A is a linear ordering $\leq\ \subseteq A \times A$ whose strict part $x < y$ is well-founded. A set A is wellorderable if there exists some wellordering $\leq$ of A.

(6a) Prove in ZF (without the Axiom of Choice) that if R is well-founded, then there exists a unique function

$$d_R : A \to V \text{ such that for all } x \in A,\ d_R(x) = \Big\{d_R(y) : R(y, x)\Big\},$$

often called *Mostowski map of R*. (It is enough here to quote correctly some appropriate, general theorem which justifies this recursive definition.)

(6b) Prove in ZF + Axiom of Foundation (without the Axiom of Choice) that if B is transitive and wellorderable, then there is a well-founded relation $R \subseteq \lambda \times \lambda$ on some ordinal number λ such that

$$B = d_R[\lambda] = \Big\{d_R(\eta) : \eta \in \lambda\Big\}.$$

(6c) Work in ZF + Axiom of Foundation. Prove that if the powerset $\mathcal{P}(A)$ of every wellorderable set is wellorderable, then the Axiom of Choice is true. *Hint*: Show by ordinal induction that every "partial universe" V_ξ is wellorderable. Part **(6b)** is used to deal with the case of limit ξ.

Problem 7. Consider the two formulas which can be used to define well-founded relations in set theory:

$$\mathsf{wf}_1(R) \iff \mathsf{Relation}(R)\ \&(\forall X)[\emptyset \neq X \to (\exists x \in X)(\forall y \in X)\neg(y, x) \in R],$$

$$\mathsf{wf}_2(R) \iff \mathsf{Relation}(R)\ \&(\exists f : \mathsf{Field}(R) :\to \mathsf{Ordinals}) \\ (\forall x, y \in \mathsf{Field}(R))[(x, y) \in R \to f(x) \in f(y)]$$

where $\mathsf{Relation}(R)$, $\mathsf{Function}(f)$, $\mathsf{Field}(R)$ and the class of ordinal numbers are defined as usual. The equivalence of these two formulas is a basic result of ZF (without the Axiom of Choice) and it implies the ZF-absoluteness of the well-founded relation, i.e., if M is a transitive set (or class) and the structure $(M, \in\restriction M)$ satisfies some specified, finite subset of the axioms of ZF, then for every relation $R \in M$,

$$R \text{ is well-founded} \iff (M, \in\restriction M) \models \mathsf{wf}_1[R] \iff (M, \in\restriction M) \models \mathsf{wf}_2[R].$$

Let ω_1 be the first uncountable ordinal.

(7a) Prove that the set

$$A = \Big\{\alpha \in \omega_1 : L_\alpha \models (\forall R)[\mathsf{wf}_1(R) \leftrightarrow \mathsf{wf}_2(R)]\Big\}$$

is closed and unbounded in ω_1.

(7b) Prove that for every ordinal α,

$$L_{\alpha+1} \not\models (\forall R)[\mathsf{wf}_1(R) \leftrightarrow \mathsf{wf}_2(R)].$$

Problem 8. A function $\pi : \mathbb{N} \rightarrowtail\!\!\!\!\twoheadrightarrow \mathbb{N}$ is a *provably in* PA *recursive permutation of* $\mathbb{N}$ if $f = \varphi_e$ for some e such that

$$\mathsf{PA} \vdash (\forall x)(\exists y)\mathsf{T}_1(e, x, y) \,\&(\forall w)(\exists! x)[(\exists y)\mathsf{T}_1(e, x, y) \,\&\, U(y) = w],$$

with the usual notation. Let

$$\mathrm{Th}(\mathsf{PA}) = \{\ulcorner\theta\urcorner : \mathsf{PA} \vdash \theta\}, \quad \mathrm{Th}(\mathsf{ZFC}) = \{\ulcorner\varphi\urcorner : \mathsf{ZFC} \vdash \varphi\}$$

where θ, φ are sentences in the relevant signatures and $\ulcorner\theta\urcorner, \ulcorner\varphi\urcorner$ are their Gödel numbers. Prove that there is a provably recursive in PA permutation of $\mathbb{N}$ such that

$$e \in \mathrm{Th}(\mathsf{ZFC}) \iff \pi(e) \in \mathrm{Th}(\mathsf{PA}).$$
