Qualifying Examination
LOGIC
Fall 2012

Please answer all questions.

You may (and you will need to) use some of the "big" theorems of logic (the Gödel Completeness and Incompleteness Theorems, the Omitting Types Theorems, Tarski's Theorem, Kleene's Normal Form Theorem, the Condensation Lemma, etc.), and when you do, make sure you quote them correctly.

You may also assume that Peano arithmetic (PA) is sound (i.e., its theorems are all true in its standard interpretation) and that Zermelo-Fraenkel Set Theory with Choice (ZFC) is consistent.

For any axiomatizable theory T and any sentence θ in the language of T, $\Box_T\theta$ is the sentence in the language of PA which naturally expresses the relation $T \vdash \theta$, and $\mathsf{Consis}(T)$ is the sentence in the language of PA which naturally expresses the consistency of T.

Problem 1. Let Ψ_n be the set of ZFC axioms with Replacement and Comprehension restricted to formulas with no more than n quantifiers. Prove that for each n, there is an $m > n$ such that $\Psi_m \vdash \mathsf{Consis}(\Psi_n)$.

Problem 2. For each of the following formulas, determine whether it is ZFC-absolute, i.e., absolute for all transitive models of some finite fragment of ZFC. Prove your answers.

(2a) "x is an ordinal and $y \in L_x$".

(2b) "x is a cardinal".

(2c) "x is an ordinal and y is a club (closed, unbounded set) in x".

(2d) "x is an ordinal and y is stationary in x" (i.e., y has a non-empty intersection with every club in x.)

Problem 3. Let $\varphi_e : \mathbb{N} \rightharpoonup \mathbb{N}$ be the recursive partial function with code e and let F be the set of codes of total recursive functions,

$$F = \{e : (\forall n)[\varphi_e(n)\downarrow]\}.$$

(3a) Prove that there is a least set $A \subseteq \mathbb{N}$ such that

$$1 \in A, \quad (\forall e)\Big([e \in F \,\&(\forall n)[\varphi_e(n) \in A]] \implies 2^e \in A\Big).$$

(3b) Prove that for this set A,

$$2^e \in A \implies e \in F \,\&(\forall n)[\varphi_e(n) \in A].$$

(3c) Define for each $a \in A$ a total, recursive function $f_a : \mathbb{N} \to \mathbb{N}$ so that the following conditions hold:

(1) For all x, $f_1(x) = x + 1$.
(2) If $a = 2^e \in A$, then for all n and all but finitely many x,

$$f_{\varphi_e(n)}(x) < f_a(x).$$

Problem 4. With $W_e = \{x : \varphi_e(x)\downarrow\}$ = the recursively enumerable set with code e, as usual, classify in the arithmetical hierarchy the relation

$$P(e, m) \iff (W_e, W_m) \text{ is a partition of } \mathbb{N} \text{ into two infinite pieces}$$
$$\iff W_e, W_m \text{ are infinite} \,\&\, W_e \cap W_m = \emptyset \,\&\, W_e \cup W_m = \mathbb{N}.$$

Problem 5. For each of the following claims, determine whether it is true or false and prove your answer.

(5a) For every consistent, axiomatizable theory T in the language of PA which extends PA and every sentence θ,

$$\text{if } T \vdash \theta, \text{ then } T \vdash \Box_T\theta.$$

(5b) For every consistent, axiomatizable theory T in the language of PA which extends PA and every sentence θ,

$$\text{if } T \vdash \Box_T\theta, \text{ then } T \vdash \theta.$$

Problem 6. For each number n, let Δn be a canonical closed term (numeral) which denotes n in PA, and for each sentence θ in the language of PA, let $(\theta)^*$ be its natural translation into the language of ZFC, namely the sentence obtained from θ by bounding all quantifiers by ω, and replacing all instances of S, $+$, and $\cdot$ with the natural ZFC formulas that define them.

For each of the following two claims, determine whether it is true or false for every formula $\varphi(v)$ with one free variable in the language of PA. You must prove your answers.

(6a) $\mathsf{ZFC} \vdash (\forall n \in \omega)\Big(\Box_{\mathsf{PA}}\varphi(\Delta n)\Big)^* \to \Big(\forall v \varphi(v)\Big)^*$.

(6b) $\mathsf{ZFC} \vdash (\forall n \in \omega)\Big(\Box_{\mathsf{ZFC}}(\varphi(\Delta n)^*)\Big)^* \to \Big(\forall v \varphi(v)\Big)^*$.

Problem 7. Let $\mathcal{L}$ be a language containing only a binary relation symbol E.

(1) Define a set Σ of $\mathcal{L}$-sentences whose models are exactly the $\mathcal{L}$-structures $\mathbf{M} = (M, E^{\mathbf{M}})$ such that $E^{\mathbf{M}}$ is an equivalence relation on M with exactly one equivalence class of cardinality n for each natural number $n > 0$.
(2) How many countable models does Σ have up to isomorphism?
(3) How many models of cardinality $\aleph_1$ does Σ have up to isomorphism?
(4) Is Σ complete? If your answer is "yes," give a proof, if not, specify two non-elementarily equivalent models of Σ.

Problem 8. Let $(A, 0, +)$ be a non-trivial abelian group, and let $R^{\mathbf{A}}$ be the relation on A defined by

$$R^{\mathbf{A}}(x, y, z, w) \quad \Longleftrightarrow \quad x + y = z + w.$$

Show that the addition map

$$(x, y) \mapsto x + y \colon A \times A \to A$$

is definable in the structure $A = (A, R^A)$ from the parameter 0, but is not definable without parameters.
