Qualifying Examination
LOGIC
Spring 2012

Try to answer all questions.

You may (and you will need to) use some of the "big" theorems of logic (the Gödel Completeness and Incompleteness Theorems, the Omitting Types Theorems, Tarski's Theorem, Kleene's Normal Form Theorem, the Condensation Lemma, etc.), and when you do, make sure you quote them correctly.

You may also assume that Peano arithmetic (PA) is sound (i.e., its theorems are all true in its standard interpretation) and that Zermelo-Fraenkel Set Theory with Choice (ZFC) is consistent.

For a formula φ in the language of PA, $\#(\varphi)$ is the Gödel number of φ; $\ulcorner\varphi\urcorner = \Delta\#(\varphi)$ is the numeral which names this number; and $\mathbf{Provable}_{\mathsf{PA}}(\ulcorner\varphi\urcorner)$ is the natural formalization in PA of "φ is provable in PA".

Two of the problems are about trees. For the purposes of this exam, a *tree* on a set X is any set $T \subseteq X^{<\omega}$ of finite sequences from X (the *finite branches* of T) which is closed under initial segments,

$$(u_0, \dots, u_{m-1}, u_m, \dots, u_n) \in T \implies (u_0, \dots, u_{m-1}) \in T.$$

An (infinite) *branch* of T is any function $f : \omega \to X$ such that for all n, $(f(0), \dots, f(n-1)) \in T$.

Problem 1. A *proof system* (for arithmetic) is any set $\mathcal{P}$ of non-empty finite sequences such that

$$(u_0, \dots, u_n, \theta) \in \mathcal{P} \implies u_0, \dots, u_n \in \omega \text{ and } \theta \text{ is a sentence in the language of PA},$$

and the set of codes

$$\text{codes}(\mathcal{P}) = \{\langle u_0, \dots, u_n, \#(\theta)\rangle : (u_0, \dots, u_n, \theta) \in \mathcal{P}\}$$

is recursive. We write

$$\mathcal{P} \vdash \theta \iff (\exists u_0, \dots, u_n)[(u_0, \dots, u_n, \theta) \in \mathcal{P}],$$

A proof system $\mathcal{P}$ is

- *sound* (for the standard model), if $\mathcal{P} \vdash \theta \implies (\omega, 0, 1, +, \cdot) \models \theta$;
- *consistent*, if there is no θ such that $\mathcal{P} \vdash \theta$ and $\mathcal{P} \vdash \neg\theta$;
- *complete*, if for every θ, either $\mathcal{P} \vdash \theta$ or $\mathcal{P} \vdash \neg\theta$.

You must prove your answers to each of the following questions:

(1a) True or False: there is a complete and sound proof system.

(1b) True or False: there is a complete and consistent proof system.

(1c) True or False: there is a complete and consistent proof system which extends Peano arithmetic, i.e.,

$$\mathsf{PA} \vdash \theta \implies \mathcal{P} \vdash \theta.$$

Problem 2. The Fixed Point Lemma (for arithmetic) says that if $\chi(v_0)$ is any formula in which only v_0 can occur free, then there is a sentence φ such that

$$\mathsf{PA} \vdash \varphi \leftrightarrow \chi(\ulcorner\varphi\urcorner).$$

(2a) Prove the following generalization of the Fixed Point Lemma:

For any two formulas of arithmetic $\chi_1(v_0, v_1), \chi_2(v_0, v_1)$ in which only v_0, v_1 can occur free, there are sentences φ, ψ of arithmetic such that

$$\mathsf{PA} \vdash \varphi \leftrightarrow \chi_1(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner), \qquad \mathsf{PA} \vdash \psi \leftrightarrow \chi_2(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner).$$

Hint. This is (apparently) not a simple Corollary of the Fixed Point Lemma, but it can be easily proved by adapting the proof of the Fixed Point Lemma.

(2b) Suppose φ, ψ are sentences of PA such that

$$\mathsf{PA} \vdash \varphi \leftrightarrow \neg\mathbf{Provable}_{\mathsf{PA}}(\ulcorner\psi\urcorner),$$
$$\mathsf{PA} \vdash \psi \leftrightarrow \mathbf{Provable}_{\mathsf{PA}}(\ulcorner\varphi\urcorner).$$

Determine whether φ is provable (in PA), true but unprovable or false, and similarly for ψ.

Problem 3. Suppose $\leq$ is a recursive wellordering of the set of natural numbers ω. Prove that we can assign to each $n \in \omega$ a theory T_n in the language of PA, such that

(1) If n_0 is the $\leq$-least member of ω, then $T_{n_0} = \mathsf{PA}$.
(2) Each T_n is axiomatizable and sound (for the standard model).
(3) The theories increase in strength as n increases in the given wellordering, i.e.,

$$n < m \implies \{\theta : T_n \vdash \theta\} \subsetneq \{\theta : T_m \vdash \theta\}$$

Hint: Use the 2nd Recursion Theorem to find a number e such that the result holds with

$$T_n = \{\theta : \varphi_e(n, \#(\theta)) = 0\}.$$

Problem 4. Let $\mathcal{L}$ be a countable language. An $\mathcal{L}$-structure $\mathfrak{M}$ is *ω-saturated* if $\mathfrak{M}$ realizes every type over every finite subset of M. $\mathfrak{M}$ is *ω-homogeneous* if, for any two tuples $\vec{a}$ and $\vec{b}$ with $\text{type}_{\mathfrak{M}}(\vec{a}) = \text{type}_{\mathfrak{M}}(\vec{b})$ and any $c \in M$, there is a $d \in M$ such that $\text{type}_{\mathfrak{M}}(\vec{a}, c) = \text{type}_{\mathfrak{M}}(\vec{b}, d)$

(4a) Prove that every ω-saturated $\mathcal{L}$-structure is ω-homogeneous.

(4b) Prove that every countable $\mathcal{L}$-structure has a countable ω-homogeneous elementary extension.

(4c) Prove or give a counterexample: *every countable $\mathcal{L}$-structure has a countable ω-saturated elementary extension.*

(4d) Prove that if $\mathfrak{M}$ is countable, ω-homogeneous and $\text{type}_{\mathfrak{M}}(\vec{a}) = \text{type}_{\mathfrak{M}}(\vec{b})$, then there is an automorphism of $\mathfrak{M}$ which takes $\vec{a}$ to $\vec{b}$.

Problem 5. For any tree T on a set A, let

$$\mathfrak{A}_T = (A, T_1, T_2, T_3, \ldots)$$

where each T_n is the set of branches of T of length n,

$$T_n(x_0, \ldots, x_{n-1}) \iff (x_0, \ldots, x_{n-1}) \in T.$$

(5a) Prove that if A is countable and T has arbitrarily long finite branches, then $\mathfrak{A}_T$ has an elementary extension $\mathfrak{M}$ in which

$$T^{\mathfrak{M}} = \{\emptyset\} \cup \bigcup_n T_n^{\mathfrak{M}}$$

has an infinite branch.

(5b) Suppose A is countable and for every n,

$$\mathfrak{A}_T \models (\forall x_0)(\exists y_0) \cdots (\forall x_n)(\exists y_n) T_n(x_0, y_0, \ldots, x_n, y_n).$$

Prove that $\mathfrak{A}_T$ has an elementary extension $\mathfrak{M}$ such that

$$(*) \qquad \mathfrak{A}_T \models (\forall x_0)(\exists y_0)(\forall x_1)(\exists y_1) \cdots \textstyle\bigwedge_n T_n(x_0, y_0, \ldots, x_n, y_n).$$

(It is part of the problem to define precisely the meaning of the infinitary sentence in $(*)$.)

(5c) Prove that the relation

$$I(T) \iff T \text{ has an infinite branch}$$

is ZFC-absolute; i.e., for some finite subset Φ of the axioms of ZFC and every transitive set or class M which satisfies Φ, if $T \in M$, then

$$M \models I(T) \iff V \models I(T)$$

where V is the class of all sets.

Problem 6. A tree T on a set X is *finitely splitting* if for every sequence $(u_0, \ldots, u_{m-1}) \in T$,

$$\{x \in X : (u_0, \ldots, u_{m-1}, x) \in T \text{ is finite}\}.$$

(6a) Prove that every infinite, finitely splitting tree has an infinite branch.

A tree T on $\{0, 1\}$ is recursive if the set of sequence codes

$$\{\langle u_0, \ldots, u_{n-1}\rangle : (u_0, \ldots, u_{n-1}) \in T\}$$

(relative to a standard, recursive coding of finite sequences) is recursive.

(6b) Prove that every infinite recursive tree on $\{0, 1\}$ has a Δ^0_2 infinite branch.

(6c) Construct an infinite, recursive tree on $\{0, 1\}$ which has no recursive infinite branch.

Problem 7. A formula of set theory is Σ_2 if it is of the form

$$\varphi \equiv (\exists y_1)(\exists y_2)\cdots(\exists y_n)\psi$$

where ψ is Δ_0, i.e., it only has bounded quantifiers $(\exists t \in w), (\forall t \in w)$.

Let α be an ordinal number such that $L_\alpha \prec_{\Sigma_2} L$, i.e., such that, for all Σ_2 formulas $\varphi(x_1, \ldots, x_n)$,

$$\forall x_1 \in L_\alpha \ldots \forall x_n \in L_\alpha[\varphi^{L_\alpha}(x_1, \ldots, x_n) \leftrightarrow \varphi^L(x_1, \ldots, x_n)].$$

Prove that all the axioms of ZFC except perhaps Replacement are true in L_α.
