

Please answer all questions. You must prove all your answers, even when this is not explicitly requested. In each problem, the level of details you give and your choice of which standard results to prove and which to use without proof should be appropriate to the question; you have to demonstrate that you know the arguments relevant to the question.

**Problem 1.** Let  $\kappa$  be an infinite cardinal of  $L$ . Let  $\alpha$  be an ordinal with  $\kappa < \alpha < (\kappa^+)^L$ . Let  $\beta \geq \alpha$  be least so that  $L_{\beta+1} \models \text{“}\exists f: \kappa \rightarrow \alpha \text{ onto”}$ . Suppose  $\beta$  is a limit ordinal. You may also assume that  $L_\beta$  satisfies your favorite finite fragment of ZFC. Prove that  $L_{\beta+1} \models \text{“}\exists g: \kappa \rightarrow \beta \text{ onto”}$ .

**Problem 2.** Which of the following is absolute for transitive models  $M$  of large enough finite fragments of ZFC that satisfy the indicated condition?

(2a) “ $x = y \cdot z$  (ordinal multiplication)”, with no further condition on  $M$ .

(2b) “ $x$  is countable”, for  $M$  so that  $(\aleph_1)^M = \aleph_1$ .

(2c) “ $x \in \bigcap \{y \mid \varphi(y, z)\}$ ” where  $\varphi$  is  $\Sigma_1$ , with no further condition on  $M$ .

**Problem 3.** Let  $T$  be a complete consistent theory in a countable language, and let  $\Sigma$  be a non-principal type of  $T$ . Prove that  $T$  has a countable model  $\mathfrak{A}$  in which  $\Sigma$  is realized infinitely many times.

**Problem 4.** Let  $\mathcal{L}$  be a countable language, and let  $\mathcal{L}^*$  be its expansion obtained by adding new distinct constant symbols  $\{c_\xi \mid \xi < \omega_1\}$ . Let  $\mathfrak{A}$  be a model of  $\mathcal{L}^*$ , with  $A - \{(c_\xi)^{\mathfrak{A}} \mid \xi < \omega_1\}$  countable. Suppose  $\mathfrak{A}$  is atomic (meaning that the only types realized in  $\mathfrak{A}$  are principal). Prove that there is a non-trivial automorphism of  $\mathfrak{A}$  iff there are  $a, b \in A$  so that  $a \neq b$  and  $a$  and  $b$  have the same type.

**Problem 5.** Let  $\{\varphi_n(x) \mid n < \omega\}$  be a recursively enumerable set of  $\Sigma_1$  formulas of PA. Suppose that for every  $n < \omega$ , there exists some  $u \in \mathbb{N}$  so that  $(\mathbb{N}; 0, 1, +, \cdot, S) \models (\varphi_0 \wedge \cdots \wedge \varphi_n)[u]$ . Let  $\mathfrak{A}$  be a non-standard model of PA. Prove that  $\{\varphi_n(x) \mid n < \omega\}$  is realizable in  $\mathfrak{A}$ , meaning that there is  $u \in A$  so that for every  $n$ ,  $\mathfrak{A} \models \varphi_n[u]$ .

**Problem 6.** Suppose  $t(n)$ ,  $g(n)$  and  $h(u, v)$  are recursive partial functions.

(6a) Prove that there is a recursive partial function  $f$  such that

$$(**) \quad f(n) = \begin{cases} g(n) & \text{if } t(n) = 0, \\ h(f(2n), f(4n)) & \text{if } t(n) \downarrow, t(n) > 0 \text{ and } f(3n) = 0, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

(6b) Prove that there is a recursive partial function  $f$  satisfying (\*\*) which is least, i.e., such that for any  $f'$  satisfying (\*\*),  $f' \supseteq f$ .

**Problem 7.** For any  $A \subseteq \mathbb{N}$ ,

$$\text{Th}(A) = \{\theta : \theta \text{ is a sentence of PA \& } \#(\theta) \in A\}$$

where  $\#(\theta)$  is the Gödel number of  $\theta$ . Recall that  $W_e$  denotes the domain of the partial recursive function coded by  $e$ . For each  $e$ ,  $\text{Consis}(e)$  is the sentence of PA which expresses the proposition that  $\text{Th}(W_e)$  is consistent.

(7a) Classify in the arithmetical hierarchy the relation

$$C(e) \iff \text{PA} \cup \text{Th}(W_e) \text{ is consistent.}$$

(7b) Prove that there exists a number  $e$  such that

$$\text{Th}(W_e) = \text{PA} \cup \{\text{Consis}(e)\}.$$

(7c) Suppose  $e$  satisfies (7b): can the theory  $\text{Th}(W_e)$  be consistent?

**Problem 8.** A total function  $f$  is **provably recursive** (in PA) if there is a  $\Sigma_1$  formula  $\varphi(v_1, \dots, v_n, w)$  which numeralwise represents it, i.e.,

$$f(x_1, \dots, x_n) = w \iff (\mathbb{N}; 0, 1, +, \cdot, S) \models \varphi(\Delta x_1, \dots, \Delta x_n, \Delta w)$$

(where  $\Delta n$  is the numeral denoting  $n$ ) and in addition

$$(**) \quad \text{PA} \vdash (\forall \vec{v})(\exists! w)\varphi(\vec{v}, w).$$

(This is equivalent to various other definitions that can be given.)

(8a) Prove that there is a total recursive function which is not provably recursive in PA.

(8b) Prove that the class of provably recursive functions is closed under composition, i.e., the scheme  $f(\vec{x}) = g(h_1(\vec{x}), \dots, h_m(\vec{x}))$ .

(8c) Is this class closed under primitive recursion? This is the scheme  $f(0) = q, \quad f(x+1) = h(f(x), x)$ .

(8d) Is this class closed under minimization? This is the scheme  $f(\vec{x}) = \mu y[g(y, \vec{x}) = 0]$  where it is assumed that  $(\forall \vec{x})(\exists y)[g(\vec{x}, y) = 0]$ .