Qualifying Examination

LOGIC

Winter 2013

DO ALL EIGHT PROBLEMS.

1. Let $\mathcal{L}$ be a language whose only symbol is a unary function symbol f. For each $\mathcal{L}$-sentence σ, let $\mathrm{Spec}(\sigma)$ be the spectrum of σ, i.e., the set of cardinalities of finite models of σ. (The *cardinality* of an $\mathcal{L}$-structure is the cardinality of its underlying set.) Let $a \geq 0$ and $b \geq 1$ be integers. Give an $\mathcal{L}$-sentence σ such that $\mathrm{Spec}(\sigma)$ is the set of positive integers of the form $a + bn$ with $n = 0, 1, 2, \ldots$.

2. Let $\mathcal{L}$ be a language, and let $\mathfrak{C}$ be a class of $\mathcal{L}$-structures. The *theory* of $\mathfrak{C}$ is the set $\mathrm{Th}(\mathfrak{C})$ of $\mathcal{L}$-sentences σ with $\mathbf{A} \models \sigma$ for all $\mathbf{A} \in \mathfrak{C}$, and the *asymptotic theory* of $\mathfrak{C}$ is the set $\mathrm{Th}_a(\mathfrak{C})$ of $\mathcal{L}$-sentences σ with $\mathbf{A} \models \sigma$ for all but finitely many $\mathbf{A} \in \mathfrak{C}$.

(a) Suppose that $\mathfrak{C}$ consists of finite $\mathcal{L}$-structures and contains only finitely many structures of cardinality n, for each natural number n. Show that the models of $\mathrm{Th}_a(\mathfrak{C})$ are exactly the infinite models of $\mathrm{Th}(\mathfrak{C})$.

(b) Suppose $\mathcal{L}$ is the empty language (i.e., no constant, function, or relation symbols), and $\mathfrak{C}$ is the class of $\mathcal{L}$-structures whose underlying set is of the form $\{1, \ldots, n\}$ for some integer $n \geq 1$. Show that $\mathrm{Th}_a(\mathfrak{C})$ is complete.

3. Let T be a countable consistent set of sentences in some language $\mathcal{L}$, and consider the space $S_n(T)$ of complete n-types of T (over $\emptyset$) equipped with the Stone topology. (The basic open sets are those of the form $\{\Sigma(\vec{x}) \mid \varphi(\vec{x}) \in \Sigma(\vec{x})\}$ for formulas $\varphi(\vec{x})$.) Let X be a subset of $S_n(T)$ which is meager, i.e., X is the union of countably many sets whose closure has no interior. Show that X can be omitted in T (i.e., there is a model of T which omits each $p \in X$).

4. Classify $\{(a, b) \mid W_a \subseteq W_b\}$ in the arithmetical hierarchy.

5. Fix a Gödel numbering of the language of PA such that each natural number n is the Gödel number of a sentence σ_n. For each n, let C be a recursive function such that $\sigma_{\mathrm{C}(n)}$ is the sentence $\mathrm{Consis}(n)$ expressing the consistency of Th(n), the set of consequences of $\{\sigma_m \mid m \in W_n\}$. Prove that there is an n such that Th(n) is the set of consequences of $\mathrm{PA} \cup \{\mathrm{Consis}(n)\}$.

6. Let σ be a sentence of the language of set theory. Assume that there is a primitive recursive function f such that

$$\begin{aligned} \text{ZFC} \vdash (\forall n)(&\text{there is a countable, transitive model of } \text{ZFC}_{f(n)} \\ &\rightarrow \text{there is a countable, transitive model of } \text{ZFC}_n \cup \{\sigma\}). \end{aligned}$$

Here ZFC_n is the set of the first n axioms of ZFC. Explain how the arithmetical formalization of

$$\text{If ZFC is consistent, then so is } \text{ZFC} \cup \{\sigma\}$$

is provable in Peano Arithmetic.

Hint: Use the formalization of the fact that the instances of Reflection are provable in ZFC.

7. Assume $V = L$.

(a) Is uncountability absolute for uncountable transitive models of ZFC_n for sufficiently large n?

(b) Is having uncountable cofinality absolute for uncountable transitive models of ZFC_n for sufficiently large n?

Hint: For (b), use Reflection and an elementary chain to get a model of size $\aleph_1$ whose ω_2 has cofinality ω.

8. Prove that $2^{\aleph_0} \neq \aleph_\omega$.

Hint: Let $X = \{f \mid f : \omega \rightarrow \omega_\omega\}$. Show that if $2^{\aleph_0} \geq \aleph_\omega$ then $2^{\aleph_0} \geq \text{card}(X)$. Then prove that $\text{card}(X) > \aleph_\omega$. To do this, assume that h is a surjection of ω_ω onto X and get a contradiction. (If you choose to use König's Theorem rather than following the hint, then don't just state König's Theorem. Prove it.)