Qualifying Examination
LOGIC
Fall 2014

Please answer all questions. You must prove all your answers, even when this is not explicitly requested. In each problem, the level of details you give and your choice of which standard results to prove and which to use without proof should be appropriate to the question; you have to demonstrate that you know the arguments relevant to the question. *Write your UID number on every page of your solution!*

Problem 1. Let L^∞ be the extension of first order logic with the "*exists infinitely many*" quantifier: i.e., we add to the formation rules for formulas the clause

if φ is a formula and v a variable, then $(\exists^\infty v)\varphi$ is also a formula,

and we interpret these formulas on a structure $\mathbf{A}$ by adding to the satisfaction relation the clause

$$\mathbf{A}, \pi \models (\exists^\infty v)\varphi \iff \text{for infinitely many } x \in A,\ \mathbf{A}, \pi\{v := x\} \models \varphi.$$

The basic notions of definability in L^∞, L^∞-elementary substructures and extensions, etc., are defined in the obvious way.

For each of the following three propositions, determine whether they are true or false and prove your answer.

(1a) In the standard structure $\mathbf{N} = (\mathbb{N}, 0, 1, +, \cdot)$ of arithmetic, every L^∞-definable relation $P \subseteq \mathbb{N}^n$ is first-order definable in $\mathbf{N}$.
(1b) If $\mathbf{N}^*$ is a proper elementary extension of the standard structure $\mathbf{N}$, then every L^∞-definable relation $P \subseteq (\mathbb{N}^*)^n$ is first-order definable in $\mathbf{N}^*$.
(1c) If $\mathbf{A}, \mathbf{B}$ are structures of the same signature and $\mathbf{B}$ is an elementary extension of $\mathbf{A}$, then $\mathbf{B}$ is an L^∞-elementary extension of $\mathbf{A}$.

Problem 2. Let $P(e, t)$ be a semirecursive (Σ^0_1) relation on $\mathbb{N}$. For each of the following two propositions, determine whether it is true or not and prove your answer.

(2a) There is a recursive partial function $f\colon \mathbb{N} \rightharpoonup \mathbb{N}$ such that

$$(\exists t)P(e,t) \implies \Big(f(e)\!\downarrow\ \&\ P\big(e, f(e)\big)\Big).$$

(2b) There is a recursive partial function $g\colon \mathbb{N} \rightharpoonup \mathbb{N}$ such that

$$(\exists t)\neg P(e,t) \implies \Big(g(e)\!\downarrow\ \&\ \neg P\big(e, g(e)\big)\Big).$$

Problem 3. Let PA be Peano arithmetic and for any sentence θ in the language of PA, let $\Box\,\theta$ be the (naturally constructed) sentence which expresses that θ is provable, so that in particular,

$$\mathbf{N} = (\mathbb{N}, 0, 1, +, \cdot\,) \models \Box\,\theta \quad \Longleftrightarrow \quad \mathsf{PA} \vdash \theta.$$

For each of the following sentences, determine whether it is true or not in the standard model, for all choices of θ, and prove your answer. You may appeal to any of the basic results of logic established in 220, provided you quote it correctly.

(3a) $(\Box\,\theta \to \theta) \to \Box\,\theta$.
(3b) $\Box\,(\Box\,\theta \to \theta) \to \Box\,\theta$.
(3c) $(\Box\,\theta \to \theta) \to \Box\,\Box\,\theta$.
(3d) $\Box\,(\Box\,\theta \to \theta) \to \Box\,\Box\,\theta$.

Problem 4. Let $\varphi_m \colon \mathbb{N} \rightharpoonup \mathbb{N}$ be the recursive partial function with code m, let $F = \{m : \varphi_m \text{ is a total function}\}$ and for any $e \in \mathbb{N}$, $S \subseteq \mathbb{N}$, set

$$P(e, S) \iff e = 1 \vee (\exists m)\Big(e = 2^m \;\;\&\;\; m \in F \;\;\&\;\; (\forall t)[\varphi_m(t) \in S]\Big)$$

The relation $P(e, S)$ defines an operator $\Phi \colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$,

$$\Phi(S) = \big\{e : P(e, S)\big\},$$

and we set

$$\overline{S} = \bigcap \big\{S : \Phi(S) \subseteq S\big\}.$$

(4a) Prove that Φ is *monotone*, i.e., for $S, S' \subseteq \mathbb{N}$,

$$S \subseteq S' \implies \Phi(S) \subseteq \Phi(S').$$

(4b) Prove that $\overline{S}$ is *the least fixed point* of Φ, i.e.,

$$\Phi(\overline{S}) = \overline{S}, \quad (\forall S')[\Phi(S') = S' \implies \overline{S} \subseteq S'].$$

(4c) Prove that $\overline{S}$ is Π^1_1, i.e.,

$$e \in \overline{S} \iff (\forall \alpha) Q(e, \alpha),$$

where α ranges over the set $\mathcal{N} = (\mathbb{N} \to \mathbb{N})$ of unary functions on $\mathbb{N}$ (Baire space) and $Q(e, \alpha)$ is arithmetical.

Problem 5. A **directed set** is a pair $(I, \leq)$ where I is a non-empty set and $\leq$ is a reflexive and transitive binary relation on I (that is, a preorder), with the additional property that every pair of elements of I has an upper bound in I: for all $i_1, i_2 \in I$ there is $j \in I$ with $i_1 \leq j$ and $i_2 \leq j$. Let $\mathcal{L}$ be a first-order language and let $(\mathbf{A}_i)_{i \in I}$ be a **directed system** of $\mathcal{L}$-structures, that is, a family of $\mathcal{L}$-structures, indexed by the elements of a directed set $(I, \leq)$, such that $\mathbf{A}_i \subseteq \mathbf{A}_j$ for all $i \leq j$ in I.

(5a) Show that there is a unique way to turn $A = \bigcup_{i \in I} A_i$ into an $\mathcal{L}$-structure $\mathbf{A}$ containing each $\mathbf{A}_i$ as a substructure. We call $\mathbf{A}$ the **directed union** of $(\mathbf{A}_i)$, denoted by $\varinjlim \mathbf{A}_i$.

(5b) Let $\mathbf{B}$ be an $\mathcal{L}$-structure containing each $\mathbf{A}_i$ as a substructure. Show that $\varinjlim \mathbf{A}_i \subseteq \mathbf{B}$.

(5c) Give an example of a directed system $(\mathbf{A}_i)$ of $\mathcal{L}$-structures and an $\mathcal{L}$-sentence σ such that $\mathbf{A}_i \models \sigma$ for all $i \in I$, but $\varinjlim \mathbf{A}_i \models \neg\sigma$.

(5d) Let T be an $\mathcal{L}$-theory and suppose $\mathbf{A}_i \models T$ for each $i \in I$. Show that $\varinjlim \mathbf{A}_i$ is existentially closed in some model of T. (If $\mathbf{A} \subseteq \mathbf{B}$ are $\mathcal{L}$-structures, then $\mathbf{A}$ is said to be **existentially closed** in $\mathbf{B}$ if for each existential $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ and $a \in A^n$ we have $\mathbf{B} \models \varphi(a) \Rightarrow \mathbf{A} \models \varphi(a)$.)

Problem 6. Let $\mathbf{A} = (A, f)$ where A is an infinite set and $f \colon A \to A$ is a bijection, viewed as a structure in the first-order language consisting solely of a unary function symbol for f. Determine all 1-types of $\mathbf{A}$ over the empty parameter set. Which ones of those are principal (= isolated)? (Include a proof for your assertions.)

Problem 7. Let $\mathbf{A} = (A, R)$ be a model of ZFC and suppose that $\mathbf{A}$ is non-standard, meaning that $(w, R \restriction w)$ is illfounded where $w = \{x \in A : \mathbf{A} \models x \in \omega\}$. Prove that there is an $\alpha \in A$ so that $\mathbf{A} \models$ "α is an ordinal" and so that $(U, R \restriction U) \models \mathsf{ZFC}$ where $U = \{x \in A : \mathbf{A} \models x \in V_\alpha\}$.

Problem 8. A function $f \colon \kappa \to \kappa$ is *canonical* if there is a wellordering B on κ so that

$$(\forall \alpha < \kappa) f(\alpha) = \text{ordertype}(B \restriction \alpha).$$

A function $g \colon \kappa \to \kappa$ *dominates* $f \colon \kappa \to \kappa$ if there is a club $C \subseteq \kappa$ such that

$$(\forall \alpha \in C) g(\alpha) \geq f(\alpha).$$

Assume $V = L$, let κ be a successor cardinal, and prove that there is a function $g \colon \kappa \to \kappa$ which dominates every canonical function $f \colon \kappa \to \kappa$. You may use the standard condensation results for L without proof. If you use any version of the Diamond Principle, you should state it precisely and prove it.