

Please answer all questions. You must prove all your answers, even when this is not explicitly requested. In each problem, the level of details you give and your choice of which standard results to prove and which to use without proof should be appropriate to the question; you have to demonstrate that you know the arguments relevant to the question.

Problem 1. Let \mathcal{L} be a countable language, let \mathbf{M} be an \mathcal{L} -structure, and $B \subseteq M$. We say that an \mathcal{L}_B -formula $\varphi(x)$ in the single free variable x is *algebraic over B* in \mathbf{M} if there are only finitely many $a \in M$ with $\mathbf{M} \models \varphi[a]$, and we say that an element a of M is *algebraic over B* if there is an algebraic \mathcal{L}_B -formula $\varphi(x)$ in \mathbf{M} such that $\mathbf{M} \models \varphi[a]$.

(1a) Show that if \mathbf{M} is \aleph_0 -categorical (that is, its theory is \aleph_0 -categorical) and B is finite, then \mathbf{M} has only finitely many elements which are algebraic over B .

(1b) Suppose \mathcal{L} extends the language $\mathcal{L}_{\text{Ri}} = \{0, 1, +, -, \cdot\}$ of rings. Show that there is no \aleph_0 -categorical \mathcal{L} -structure whose \mathcal{L}_{Ri} -reduct is a field.

Problem 2. Let \mathcal{L} be a language, let T be a complete \mathcal{L} -theory, let \mathcal{L}' be a language expanding \mathcal{L} , and let T' be an \mathcal{L}' -theory containing T . Assume that T has a model which has an elementary extension which can be expanded to a model of T' . Prove that every model of T has an elementary extension which can be expanded to a model of T' .

Problem 3. Work in the language of PA. As usual Δm is the term which denotes the number m , and $\varphi(\Delta m)$ is obtained from $\varphi(v)$ by replacing all free occurrences of v with Δm .

(3a) For a formula $\varphi(v)$ in the single free variable v , consider the proposition (Q_a) : if for every m , $\varphi(\Delta m)$ is provable in PA, then the sentence $(\forall v)\varphi(v)$ is true.

- (i) Is (Q_a) true (meaning true for all φ)?
- (ii) Is (Q_a) expressible in the language of PA?
- (iii) If you answered yes for (ii), is the sentence expressing (Q_a) provable in PA? is the negation of the sentence expressing (Q_a) provable in PA?

(3b) Again for a formula $\varphi(v)$ in the single free variable v , consider the proposition (Q_b) : if for every m , $\varphi(\Delta m)$ is provable in PA, then the sentence $(\forall v)\varphi(v)$ is provable in PA.

- (i) Is (Q_b) true?
- (ii) Is the sentence expressing (Q_b) in PA provable in PA? is the negation of the sentence expressing (Q_b) provable in PA?

Problem 4. A partial function $f : \mathbb{N}^k \rightarrow \mathbb{N}$ is Σ_2^0 -recursive if its graph is a Σ_2^0 relation. Prove that if h, g, τ are Σ_2^0 -recursive, then so is the partial function $f(n, x)$ defined from them by *nested recursion*: $f(0, x) = g(x)$, $f(n + 1, x) = h(f(n, \tau(x)), n, x)$.

Problem 5. A set $X \subseteq \mathbb{N}$ is an *initial segment* of \mathbb{N} if either $X = \mathbb{N}$, or $X = \emptyset$, or $X = \{0, \dots, m\}$ for some $m \in \mathbb{N}$. Classify the set $A = \{e \mid W_e \text{ is an initial segment of } \mathbb{N}\}$ in the arithmetical hierarchy.

Problem 6. Work in the language of PA, and for each sentence φ let $\# \varphi$ as usual be the Gödel number for φ . Let T be a consistent recursively axiomatizable extension of PA.

Classify the set $\{\#\varphi \mid T \vdash \varphi\}$ in the arithmetical hierarchy. Recall that this means you must find a class of the arithmetic hierarchy (Σ_n , Π_n , or Δ_n) and prove that the set belongs to this class and does not belong to any smaller class.

Problem 7. Let M_α for $\alpha \in \text{ON}$ be transitive sets and let $M = \bigcup_{\alpha \in \text{ON}} M_\alpha$. (Formally these objects are definable classes: there is a formula φ and a parameter a so that $M_\alpha = \{x \mid \varphi(\alpha, x, a)\}$ and $M = \{x \mid (\exists \alpha \in \text{ON})\varphi(\alpha, x, a)\}$.) Suppose that (i) for every $\alpha < \beta$, $M_\alpha \in M_\beta$, and (ii) for every limit λ , $M_\lambda = \bigcup_{\alpha < \lambda} M_\alpha$. Prove that if for arbitrarily large $\alpha \in \text{ON}$, M_α satisfies the Comprehension schema, then M satisfies the Comprehension schema. You may assume all axioms of ZFC hold in V .

Problem 8. Let $\alpha < \beta < \gamma$ be limit ordinals. Suppose L_γ satisfies “ α is a regular cardinal”, and “ β is the cardinal successor of α ”. Suppose $\kappa < \alpha$ and there is a cofinal increasing function $f: \kappa \rightarrow \alpha$ definable over L_γ by a Σ_1 formula with parameter $p \in L_\gamma$.

(8a) Let H be the Σ_1 Skolem hull of $\alpha \cup \{p, \alpha\}$ in L_γ . Prove that $H \supseteq \beta$.

(8b) Prove that κ , α , β , and γ all have the same cofinality in V .