Qualifying Examination
LOGIC
Fall 2016

Please answer all questions. You must prove all your answers, even when this is not explicitly requested. In each problem, the level of details you give and your choice of which standard results to prove and which to use without proof should be appropriate to the question; you have to demonstrate that you know the arguments relevant to the question.

Problem 1. Let $\mathbf{M} = (M; \dots)$ be a model of Peano Arithmetic (PA). Below, "definable" includes the possibility of parameters. Show that $\mathbf{M}$ has an elementary extension $\mathbf{M}^*$ with the following property: for every $X \subseteq M$ there is some $X^* \subseteq M^*$ such that $X^* \cap M = X$ and X^* is definable in $\mathbf{M}^*$.

Problem 2. Say that a formula φ in the language of set theory with a single free variable *defines* a set x if $\varphi(x)$ holds (in V) and for every $y \neq x$, $\varphi(y)$ fails.

(2a) Prove that any set x definable by a Σ_1-formula is countable.

(2b) Is the same true for Π_1-formulas? (Proof or counterexample.)

Problem 3.

(3a) Prove the following schema in ZFC: for every unbounded class U of ordinals, every ordinal α, and every class function $F\colon U \to \alpha$, there exists an unbounded class $U' \subseteq U$ so that F is constant on U'.

(3b) Prove that every countable model $\mathbf{A}$ of ZFC has an elementary end extension, meaning a proper elementary extension $\mathbf{B}$ so that the ordinals of $\mathbf{A}$ form an initial segment of the ordinals of $\mathbf{B}$.

Problem 4. Fix a recursive coding $\#\phi$ for sentences in the language of set theory. For each $e \in \mathbb{N}$, let T_e be the set of sentences ϕ so that $\{e\}(\#\phi) = 0$. Classify the set

$$A = \{e : T_e \text{ axiomatizes } \mathsf{ZFC}\}$$

in the arithmetical hierarchy.

Problem 5. Recall that a nonempty subset of $2^{<\omega}$ which is closed under taking initial segments is called a *tree*. An *infinite branch* of a tree T is a function $\beta\colon \omega \to 2$ such that $\big(\beta(0), \dots, \beta(n-1)\big) \in T$ for all n. It is well-known (König's Tree Lemma) that every infinite tree has an infinite branch. Fixing a recursive coding $\langle - \rangle$ of sequences of natural numbers in ω, a set $T \subseteq 2^{<\omega}$ is called recursive if the subset

$$\big\{\langle s_0, \dots, s_{n-1}\rangle : (s_0, \dots, s_{n-1}) \in T\big\}$$

of ω is recursive.

(5a) Show that there exists an infinite recursive tree without a recursive infinite branch.

(5b) Show that if an infinite recursive tree has a single infinite branch, then this branch is recursive.

(5c) Show that every infinite recursive tree has an infinite branch β such that $B \leq_{\mathrm{T}} K$; here B is the subset of ω with characteristic function $\chi_B = \beta$, and K is the halting set.

Problem 6. Assume $V = L$. Let Σ be some fixed finite fragment of ZFC, as large as you wish. Let ψ be a Δ_0-formula and let $\varphi(\delta, B)$ be the formula

$$(\exists X \subseteq \delta)\, \psi(L_\delta, B, X).$$

Let κ be a cardinal and let $A \subseteq \kappa$. Suppose that for every $\tau < \kappa$, $\varphi(\tau, A \cap \tau)$ holds. Let $\alpha_\tau < \kappa$ be least so that $\tau, A \cap \tau \in L_{\alpha_\tau}$ and $L_{\alpha_\tau} \models \Sigma \cup \{\varphi(\tau, A \cap \tau)\}$.

(6a) Prove that such α_τ exists for each $\tau < \kappa$.

(6b) Prove that all elements of L_{α_τ} are definable over L_{α_τ} using parameters in $\tau \cup \{\tau, A \cap \tau\}$.

(6c) Set $\tau \sqsubseteq \delta$ iff there exists an elementary map $\pi\colon L_{\alpha_\tau} \to L_{\alpha_\delta}$ with $\pi(\tau) = \delta$ and $\pi(A \cap \tau) = A \cap \delta$. Prove that $\sqsubseteq$ is a tree order, meaning an order so that for every δ, the set $\{\tau : \tau \sqsubseteq \delta\}$ is well-ordered by $\sqsubseteq$.

Problem 7. Let σ be a sentence in the language of arithmetic. We let $\Box\sigma$ denote the sentence $\exists x\, \mathbf{Proof}_{\mathsf{PA}}(x, \ulcorner \sigma \urcorner)$ expressing that $\mathsf{PA} \vdash \sigma$. Inductively define $\Box^n \sigma$ by

$$\Box^0 \sigma := \sigma, \qquad \Box^{n+1} \sigma := \Box(\Box^n \sigma).$$

In the following you may assume that PA is consistent. We let Δm be the term denoting the natural number m.

(7a) Show that there is a formula $\theta(x, y)$ such that for all sentences σ and all $n > 0$,

$$\mathsf{PA} \vdash \theta(\Delta n, \ulcorner \sigma \urcorner) \leftrightarrow \Box^n \sigma.$$

(7b) Is there a formula $\theta(x, y)$ with the property in (a) for all σ and all n, including $n = 0$?

Problem 8. Let $\mathcal{L}^* \supseteq \mathcal{L}$ be countable languages and $\mathbf{M}$ be a saturated $\mathcal{L}$-structure of uncountable cardinality. Let T^* be an $\mathcal{L}^*$-theory which is consistent with the $\mathcal{L}$-theory of $\mathbf{M}$. Show that then $\mathbf{M}$ can be expanded to an $\mathcal{L}^*$-structure which is a model of T^*.