Qualifying Examination
LOGIC
Spring 2016

Please answer all questions. You must prove all your answers, even when this is not explicitly requested. In each problem, the level of details you give and your choice of which standard results to prove and which to use without proof should be appropriate to the question; you have to demonstrate that you know the arguments relevant to the question.

Problem 1. Let G be a (multiplicatively written) group, viewed as a structure in the language which has a single constant symbol 1 interpreted as the identity element of G and a binary function symbol $\cdot$ interpreted as the group operation of G. For $g \in G$ we write $g^n = g \cdot g \cdots g$ (n times), so $g^0 = 1$, $g^1 = g$. One says that an element g of G has *finite order* if $g^n = 1$ for some $n \geq 1$, one says that G is a *torsion group* if every element of G has finite order, and G is *torsion-free* if the only element of finite order of G is 1.

(1a) Show that there exist elementarily equivalent groups G and H such that G is a torsion group, but H is not.

(1b) Suppose that every nonempty $\emptyset$-definable subset of the group G contains an element of finite order. Show that then G is elementarily equivalent to a countable torsion group.

(1c) Show that there is a set Σ of sentences whose models are exactly the torsion-free groups. Prove that there cannot be a finite such set Σ.

Problem 2. Let $\mathcal{L}$ be a countable first-order language and T be a complete $\mathcal{L}$-theory without finite models.

(2a) Show that the elementary substructures of atomic $\mathcal{L}$-structures as well as unions of elementary chains of atomic $\mathcal{L}$-structures are atomic.

(2b) Call an $\mathcal{L}$-structure *minimal* if it has no proper elementary substructure. Show that if T has a countable atomic model which is not minimal, then T has an atomic model of size $\aleph_1$.

(2c) Show that if $\mathbf{M}$ is a minimal model of T and $\mathbf{M}'$ is a prime model of T, then $\mathbf{M} \cong \mathbf{M}'$.

(2d) Show that if T has a prime model which is not minimal, then T has an atomic model of size $\aleph_1$.

(2e) Give an example where the hypothesis of (d) holds.

Problem 3. Assuming that ZFC is consistent, prove that it is not finitely axiomatizable.

Problem 4. Assume $V = L$.

(4a) Prove that L_{ω_2} is Σ_1 elementary in L.

(4b) Prove that there is a function $f \colon \omega_1 \to \omega_1$, Δ_1 definable over L_{ω_1}, so that for every $A \subseteq \omega_1$ and every Σ_1 formula φ, there is a club $C \subseteq \omega_1$ so that for every $\alpha \in C$, if $L \models \varphi[A]$ then $L_{f(\alpha)} \models \varphi[A \cap \alpha]$.

Problem 5.

(5a) Let $\mathfrak{A}$ be a model of $\mathsf{ZFC} - \mathsf{Powerset}$. Prove that there is a Σ_1 formula φ so that $\{x \in |\mathfrak{A}| \mid \mathfrak{A} \models \varphi[x]\}$ is not Π_1 definable (even with parameters) over $\mathfrak{A}$.

(5b) Prove that in Question (4b) one cannot strengthen the conclusion to require that $L \models \varphi[A]$ iff $L_{f(\alpha)} \models \varphi[A \cap \alpha]$.

Problem 6. Say that a function $s\colon \omega \to 2^{<\omega}$ *converges* to $x \in 2^\omega$ if for every k, $s(n) \upharpoonright k$ is eventually equal to $x \upharpoonright k$ as $n \to \infty$. (Here $2^{<\omega}$ and 2^ω are respectively the sets of finite and infinite binary sequences.)

(6a) Prove that there is a recursive $s\colon \omega \to 2^{<\omega}$ which converges to $\emptyset'$.

(6b) Let $A \subseteq \omega$ be Δ^0_2. Prove that there is a recursive $t\colon \omega \to 2^{<\omega}$ which converges to the characteristic function of A.

Problem 7. By a *permutation* we mean a bijection $\mathbb{N} \to \mathbb{N}$. ϕ_e ($e \in \mathbb{N}$) below is the standard enumeration of the recursive partial functions. Prove the following:

(7a) The set of recursive permutations forms a group Π under composition: the composition of two recursive permutations is recursive, and the inverse of every recursive permutation is recursive.

(7b) Not every permutation is recursive.

(7c) The set $P = \{e \in \mathbb{N} : \phi_e \in \Pi\}$ is not r.e.

(7d) There is no r.e. subset P' of P such that $\{\phi_e \mid e \in P'\}$ is equal to Π.

(7e) There is no r.e. subset P' of P such that $\{\phi_e \mid e \in P'\}$ generates Π. (In particular, Π is not finitely generated.)

Problem 8. Recall that $\mathcal{K} = \emptyset' = \{e \in \omega \mid \phi_e(e)\downarrow\}$. Determine which of the following are true for all m. Provability is formalized using Gödel codes in the usual way, Δm is the term denoting m, and $t \in \mathcal{K}$ for a term t is the natural formalization of the statement that the interpretation of t belongs to $\mathcal{K}$.

(8a) If $\mathsf{PA} \vdash \Delta m \in \mathcal{K}$ then $m \in \mathcal{K}$.

(8b) If $m \in \mathcal{K}$ then $\mathsf{PA} \vdash \Delta m \in \mathcal{K}$.

(8c) PA proves that if $\mathsf{PA} \vdash \Delta m \in \mathcal{K}$ then $\Delta m \in \mathcal{K}$.

(8d) PA proves that if $\Delta m \in \mathcal{K}$ then $\mathsf{PA} \vdash \Delta m \in \mathcal{K}$.

(8e) If $m \notin \mathcal{K}$ then $\mathsf{PA} \vdash \Delta m \notin \mathcal{K}$.